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MODERN PERSPECTIVE:

A TREATISE

UPON THE

PRINCIPLES AND PRACTICE OF PLANE AND CYLINDRICAL PERSPECTIVE

BY

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PREFACE.

IN compliance with the wishes of my pupils, and in fulfilment of a promise made some time since to the publishers of the *American Architect and Building News*, I have collected and revised a series of papers upon Perspective, which I five years ago contributed to the columns of that journal, adding half-a-dozen chapters and a dozen pages of illustrations.

A new treatment of so old a theme would be uncalled for, did not even the more elaborate treatises seem to be deficient in comprehensiveness and scientific simplicity, while the practical handbooks fail to make the reader acquainted with some methods that are found in experience to be among the most convenient and practical of all. Most of what I have to say is, of course, in substance, an old story; but it is a story which can, I think, be told again with profit, so as the better to lead up to the matter that is comparatively new. That I have anything to offer which is absolutely new, that I have in my explorations found any field absolutely untrodden by my predecessors, I can hardly suppose: I am too used, in these regions, to discover the footprints

of unknown or forgotten pioneers in what I had taken to be really *terra incognita*. But I am sure that if the reader will accompany me, he will come to some things that, if not absolutely novel, are new to him, and that he will reach some points of view from which the more familiar ground will present an unaccustomed aspect.

This discussion of the subject differs from that generally given, in several particulars; much greater prominence being assigned to the phenomena of parallel planes than is usual, and use being made of the laws thus established to determine the perspective of lines of intersection and of shadows, — subjects that seem hitherto to have received but little attention.

The perspective of divergent lines, also, and of shadows cast by divergent rays, as from an artificial source of light, is a subject that seems to have been almost entirely neglected.

In the course of these investigations it will appear that the horizontal plane hardly deserves the paramount importance commonly assigned to it, and that the practice of referring all constructions to that plane is productive of needless inconvenience. The well-known method, also, of points of distance, and points of measures, which is generally treated as an auxiliary method of but limited serviceability, will be shown to be of universal application, and to suffice for the solution of almost all problems. The development of this method

to its legitimate results leads to the consistent use of the Perspective Plan, rendering unnecessary the construction of the orthographic plan, by the aid of which perspective drawings are commonly made.

Any treatise on perspective is, of course, mainly directed to meet the wants of the architect; and the problems with which he deals are free from most of the perplexities that constantly annoy the student of nature. But there are difficulties and apparent anomalies which confuse the mind even of the architectural draughtsman, and in disposing of these it is possible also to explain the discrepancies which are found to exist between sketches made faithfully from nature and drawings made according to the common perspective rules,— discrepancies which have naturally produced among artists a certain disregard and contempt for the rules themselves. It will be shown, as indeed hardly needs to be pointed out, that in drawing from nature one works, virtually, not upon a plane, but upon a cylinder. The discussion of Plane Perspective needs to be supplemented, then, by chapters on Cylindrical, or, as it is sometimes called, Panoramic Perspective; and an explanation of the principles and rules of this method show its results to be exactly conformable, in kind, to those reached when drawing merely by the eye. Much that I have to say is accordingly as pertinent to the work of the landscape painter or the historical painter

as to that of the architect. Indeed, the questions that arise when the human figure is to be drawn in perspective mainly concern them.

A separate chapter discusses certain methods employed for limiting the space required for making drawings in perspective, especially that of the late M. Adhémar, — methods of the greatest value when, as in fresco-painting or scene-painting, the picture is large compared with the size of the room in which it is to be made. I have taken the liberty of considerably modifying the details of M. Adhémar's processes, and in an important particular have suggested an alternative procedure, which, if not intrinsically preferable to his own, is at least more in accord with the point of view taken in this work.

To this chapter I have added a chapter upon the interpretation of Perspective Drawings. Photography has given to the discussion of this subject an importance which it did not previously possess, for it is often desirable to obtain from the perspective view taken by the camera the real proportions or dimensions of the object shown. This is sometimes impossible, sufficient data not being furnished by the picture itself, and no other information being accessible. But when it is possible it is not difficult, as I have endeavored to make plain.

In all these chapters I have avoided a too formal method of demonstration, using a somewhat conversa-

tional style, and endeavoring to make the subject intelligible without employing the apparatus of theorems and problems. For those, however, whose tastes or habits of mind might demand a more concise and formal treatment, I have added a couple of chapters, in the first of which are collected the geometrical principles involved in the preceding pages, while the second contains, in the form of geometrical problems, solutions not only of the simple questions which the ordinary practice of perspective drawing presents, but of most of the more elementary problems of Descriptive Geometry.

Finally, to meet the practical needs of the practical draughtsman, I have added, at the end of the book, a chapter upon The Practical Problem, showing just how one goes to work to lay out the main lines of a perspective drawing according to the system presented in the previous chapters. To these the student is referred for the further illustration of matters of detail.

These last three chapters are, in a sense, quite independent of what has preceded them, and might about as well have come first as last. Some readers may find an advantage in first obtaining a general view of the whole subject by their aid, before undertaking to follow the more detailed treatment presented in the body of the work.

The plates that accompany the text have taken, altogether, quite as much time and labor in their prep-

aration as has the text itself, — a labor which has been lightened to me by the intelligent co-operation of a score of young men, most of them my pupils, who have, one after another, taken their turn at what has seemed a never-ending task. Among these I owe my acknowledgments for something more than merely clerical service to Messrs. H. F. Burr, A. H. Munsell, A. B. Harlow, G. L. Heins, F. D. Sherman, and G. T. Snelling. Especially ought I to mention the name of Mr. A. J. Boyden, to whose intelligence and skill nearly half the series bear witness.

I cannot lay down my pen without acknowledging my indebtedness in this, as in every other study that I pursued under his direction, to my friend and teacher, Professor Henry L. Eustis, of the Lawrence Scientific School. It was his cordial appreciation and sympathy that first encouraged me to pursue the path of these investigations. This was twenty-five years ago, but I have not forgotten it, and have borne in mind, as every new point has presented itself, the pleasure he would take in following my argument.

WILLIAM R. WARE.

Dec. 20, 1882.

I have taken advantage of the opportunity offered by the issue of a new edition of this book to revise the text and to add in an Appendix some particulars of interest. I have also slightly changed the notation, the chief alterations being the use of the word *horizon* instead of *trace* for the perspective of the horizon of a plane, and the use of the word *trace* for its *initial line*. W. R. W.

October 31, 1900.

MODERN PERSPECTIVE

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MODERN PERSPECTIVE.

CHAPTER I.

THE PHENOMENA OF PERSPECTIVE.

A DRAWING made *in perspective* undertakes to represent objects of the shape and size that they actually appear from a given point. It has to do only indirectly with their real shape and size, being mainly concerned with their apparent outlines and dimensions. Before trying to learn how to draw them, then, it is obviously desirable to find out how they really look. This first chapter will accordingly be taken up with considering the *appearances* of things, the *phenomena* with which perspective has to do.

The *things* in question, as always in the scientific study of form, are lines, especially straight lines; plane figures, especially rectangular figures and the circle; and solid objects, especially the sphere and cylinder. The appearance of solids bounded by plane surfaces is determined, of course, by the aspect of the plane figures that bound them.

1. Certain phenomena in regard to the shape and size of these things are sufficiently obvious. It does

not need to be pointed out that everything seems smaller — that is to say, subtends a smaller visual angle — when at a distance from the eye than when near; that consequently the more distant portions of a straight line seem smaller than equal divisions near at hand; that in rectangular figures the farther sides occupy less space to the eye than the nearer sides, so that they present, in most positions, a trapezoidal rather than a rectangular aspect, the sides inclining towards one another; that a circle when seen in perspective generally appears as an ellipse, and that the centre of the circle does not occupy the centre of the ellipse, but is nearer to the farther than to the hither edge. These *qualitative* determinations are easy enough. But it is not so easy to determine the relations of *quantity*, to tell *how much* smaller a given distance will make a given line appear, or just at *what* angle the sides of the rectangle seem inclined, and in what direction they seem to run. To determine these things with exactness is the chief object of these methods, — an object to be reached through the study of another class of phenomena, the appearances not of limited and finite lines and planes, but of lines and planes supposed to be indefinitely extended. Indeed, finite lines and planes are in perspective considered merely as portions of the indefinitely extended lines and planes in which they lie.

The station point.

2. The position of the spectator, that is to say, of the spectator's eye, is called the *Station Point*.

3. All lines lying in one and the same direction, and consequently parallel to each other, are said to belong to the same *system* of lines. Each line is an *element* of the system.

Systems of parallel lines.

Now if we imagine the lines of any system to be indefinitely extended both ways, we shall encounter the following phenomena.

4. All the lines of a system, that is, all lines parallel to each other in space, seem to converge towards two infinitely distant points. These points are called the *vanishing points* of that system of lines. They are 180° distant from each other.

Vanishing points.

The vanishing points of a line are the utmost possible limits of its apparent extension, even though infinitely extended. For a straight line, although infinitely long, cannot subtend an arc of more than 180°; it cannot seem more than a semicircle.

The beams of the sun, or the shadows of clouds, at sunset, which seem to separate overhead and converge near the opposite horizon, afford a capital instance of parallel lines with two vanishing points. So also do parallel lines of cloud, and, in streets, the lines of sidewalks, eaves, and house-tops. They appear as great circles of the sphere of which the eye is the centre.

5. Now what is very curious is that whichever element of the system one looks at seems straight, the others, on both sides, seeming concave towards it. The horizon itself, which seems straight when one looks at it, seems curved if one looks up or down. Other hori-

zontal lines, when regarded with reference to the horizon, seem parallel to it, and farthest removed from it, where they are nearest the eye, approaching it at a constantly increasing angle as they retreat towards their vanishing points.

These singular phenomena, though constantly before our eyes, are little noticed, and consequently but little known ; but they sometimes force themselves upon the draughtsman's attention, causing much confusion in his drawing and in his mind. The fact that most straight lines, all indeed except one, always seem curved is the basis of the method of curvilinear or panoramic perspective, which will form the subject of a subsequent chapter.

6. Either vanishing point of any system of lines may be found by *looking* in the direction followed by the lines of that system; the vanishing point will then be seen full in front of the eye.

That element of the system which passes through the eye, or station point, will be seen *endwise*, the line appearing as a point, coinciding with and covering the vanishing point, which is at its farther extremity. Such a line we call an Optical Line.

7. In like manner, all planes parallel to one another, and whose *axes* accordingly belong to the same system of lines, are said to belong to the same *system* of planes. Each plane is an *element* of the system. By the *axis* of a plane is meant any line at right angles, or perpendicular, to it.

8. All the planes of a system, that is, all planes parallel to each other in space, seem to converge towards an infinitely distant line, which is the limit of their utmost extension. A plane, though seemingly infinitely extended, like the sea, cannot subtend an arc of more than 180° in every direction; it cannot seem more than a hemisphere. Its limiting line accordingly will be a great circle of the infinite sphere, of which the eye, or station point, is the centre. This line is called the *horizon,* or *vanishing line* of the system of planes.

Systems of parallel planes.

Vanishing lines, *traces*, or horizons.

9. The vanishing line or horizon of any system of planes may be found by glancing along that plane of the system which passes through the eye. On looking in any direction at right angles to the axis of the system of planes, it is seen full in front of the eye. That element of the system of planes which passes through the eye is seen *edgewise,* the plane appearing as a line, covering and coinciding with the vanishing line, or horizon of the system, which is its outer extremity. Such a plane we call an Optical Plane.

Such a line is *the* Horizon, which limits at once the plane of the earth, and the plane, or hemisphere, of the sky. We may call such a line the *vanishing line* of a system of planes, just as we speak of the vanishing point of a system of lines; but as it is common to call any indefinitely extended right line a vanishing line, it is more convenient to speak of the *horizon* of a system

of planes, distinguishing the real Horizon, or vanishing line of horizontal planes, by a capital H.

10. Any point or line lying in a right line passing through the eye seems exactly to cover and coincide with the vanishing point of the system to which the line belongs. So also any line, figure, or surface, lying in a plane passing through the eye, appears as a right line, and seems to cover and coincide with a portion of the horizon of the system to which the plane belongs.

11. Vanishing points and horizons, have to do only with the *direction* of lines and planes, not with their *position.* Hence, objects whose lines and planes are parallel have the same vanishing points and horizons, whatever their position to the right or to the left, above or below the spectator.

12. A plane surface upon a solid object cannot be
Solid objects. seen unless it is on the side of the object towards the horizon of that plane.

13. It is obvious that all systems of horizontal lines have their vanishing points in the Horizon, and conversely, that the Horizon passes through the vanishing points of all systems of horizontal lines. The same is true, of course, of vertical or inclined planes, and the lines that lie in them or are parallel to them. From these considerations we can frame the following propositions, which are the fundamental propositions of our system of perspective.

(*a*) All lines, or systems of lines, lying in or parallel to a system of planes, have their vanishing points in the horizon of that system. Hence:—

The three principal Rules of perspective.

I. *A line lying in a plane has its vanishing point in the horizon of that plane.*

Conversely:

The horizon of any system of planes passes through the vanishing points of all lines parallel to them. Hence: —

II. *The horizon of a plane passes through the vanishing points of any two lines that lie in it*, that is, of any two elements of the plane.

(*b*) The horizons of all the systems of planes which can be passed through a line, or parallel to it, in any direction, pass through the vanishing point of the system to which the line belongs, and intersect each other at that point.

Conversely: —

A line, or system of lines, lying in or parallel to two planes, has its vanishing point at the intersection of their horizons. Hence: —

III. *The line of intersection of two planes has its vanishing point at the intersection of their horizons.*

Conversely, if several systems of planes are parallel to the same line, all the lines of intersection of all the planes of all the systems will be parallel to it, and have the same vanishing point.

IV. The Optical lines and planes will make the same angles with each other at the eye, or station point,

that the other lines and planes of the respective systems make with each other.

14. The reader is recommended to take the pains not only to satisfy himself of the truth of these propositions, which he will easily do, but also to verify them by examples, determining for himself, in his daily walks, at what distant points in the earth or the sky the vanishing points of different lines are to be *looked for*, lines horizontal, vertical, or inclined; and in like manner to trace the *horizons* of the different planes he encounters in roofs or walls, exemplifying these propositions over and over again until they become perfectly obvious and familiar.

The vanishing points of the eaves, for example, and of the raking cornice or other steepest line of a roof, are easily found by *looking* in the directions they pursue. These two directions determine the inclination of the plane of the roof in which they lie. Its *horizon* is a great circle, or straight line, cutting across the sky from one of these vanishing points to the other. In the case of two intersecting roofs, the vanishing point of the hip or valley that marks their intersection is found at the intersection of their horizons.

15. The discussion of a problem in perspective cannot be considered complete until the vanishing point of every line and the horizon of every plane has been determined.

CHAPTER II.

PHENOMENA RELATING TO THE PICTURE.

IN the first chapter we considered the phenomena of perspective in nature; that is to say, certain appearances of the geometrical lines and surfaces with which perspective has to do.

Let us now — deferring to the fifth chapter all question of exact magnitudes, and of the precise determination of forms — consider in like manner the principal phenomena, the main characteristics, of a perspective drawing.

In so doing we will leave all *quantitative* determinations till by and by, and assume, or guess at, any dimensions or other data we may need; or determine them by judgment, or by the eye. But as this is just the way that such data are always determined when sketching, either from nature or from the imagination, it follows that what we have now to say is specially interesting to the artist and the amateur, since it comprises almost everything that he needs in his own work.

16. The picture is supposed to be drawn upon a plane surface called the *plane of the picture*, and so drawn that if the picture were transparent, **The plane of the picture.**

every point and line of the drawing would cover and coincide with the corresponding points and lines of the objects represented, as seen from a given position, the Station Point; the plane of the picture being at a given distance and in a given direction.

17. The distance and direction of the picture are taken upon a line passing through the station point, at right angles with the plane of the picture. This line is of course an axis of that plane (4). It is called *the* Axis. If the Axis is horizontal, the picture is vertical, and this is the usual position. But if the Axis is inclined to the horizontal plane, the plane of the picture is at an angle with the vertical direction, as sometimes happens.

The Axis, Centre, and Station Point. *Perspectives.*

18. The point in the plane of the picture nearest to the eye, or Station Point, S, is called the Centre of the picture, V^C. It is the point where the Axis pierces it. The distance from the Station Point to the Centre is the length of the Axis.

The term Point of Sight is used sometimes to designate the Station Point, sometimes the Centre of the Picture. Since it is thus ambiguous, we will not employ it.

19. The representation in a perspective drawing of a point, or line, or of the vanishing point of a line or system of lines, or of the vanishing line, or horizon, of a plane or system of planes, is called the *perspective* of the point, or line, or vanishing point, or horizon.

Figure 1 represents the picture-plane, PP, as a transparent plane on which are drawn the perspectives of the lines behind it. The perspectives of the lines drawn on the vertical plane behind it, and which are consequently parallel to the plane of the picture, are parallel to the lines themselves, whether horizontal, vertical, or inclined, and though shorter, they are divided proportionally to them. The perspectives of the lines at right angles to the plane of the picture, however, differ from them in both magnitude and direction. This illustrates the following propositions.

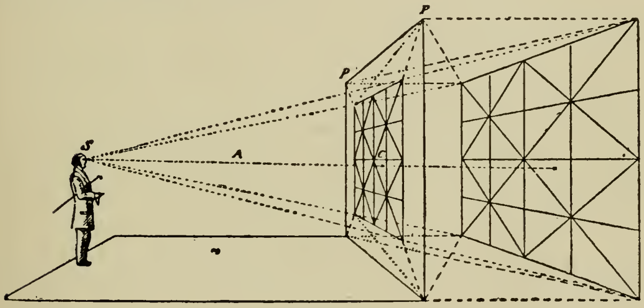

Fig. 1.

20. Lines parallel to the picture-plane, whatever their direction, have their perspectives drawn parallel to themselves; that is, in their real direction. The magnitude of the perspective of any such line is less than that of the real line, according as the distance of the line itself from the picture is greater, but its parts are proportional to the corresponding parts of the line represented. **Lines parallel to the picture.**

21. When such lines are parallel they have their

perspectives parallel to each other, and to the lines themselves.

22. The perspectives of lines not parallel to the plane of the picture are not parallel to the lines themselves, nor to each other, but are drawn converging towards a point which is the perspective of their vanishing point. In this case, as the real lines *seem* to converge towards their real vanishing point, so their perspective representations *do* converge towards the perspective of their vanishing point.

Lines inclined to the picture.

23. Hence if the picture-plane be vertical, as it usually is, the perspective of the horizontal lines that are *parallel to the plane of the picture* will be horizontal and parallel, that of inclined lines parallel to the picture will be parallel and inclined at the same angle with the lines themselves, and that of all vertical lines will be drawn vertical and parallel. All other systems of lines, whether horizontal or inclined, will have their perspectives converging to the perspective of their vanishing points.

Horizontal lines.

24. Of the two vanishing points belonging to every system of lines (4) one will, in general, be behind the spectator, and one in front of him; this last will be behind the plane of the picture, and its perspective will be somewhere in the plane of the picture and at a finite distance. But if the lines of the system in question be parallel to the plane of the picture, the perspective of both vanishing points will be at an infinite distance upon it, in opposite directions; and lines drawn to them will, of course, be parallel (20).

25. It is often asked why the apparent convergence of vertical lines is not represented by the convergence of their perspectives, just as much as that of horizontal lines. It is, just as much. For it is only those horizontal lines *which are inclined to the picture* whose perspectives are drawn to a vanishing point. The perspectives of horizontal lines parallel to the picture are drawn parallel to themselves, just as those of vertical lines are. And when the Axis is inclined so that the plane of the picture is no longer vertical, and vertical lines are no longer parallel to it, they, too, are drawn converging, one of their vanishing points, either the zenith or nadir, being now behind the picture, and its perspective at a finite distance upon it. This case will be discussed hereafter. See Plate VII.

Vertical lines.

26. Moreover, although lines parallel to the picture, whether vertical, horizontal, or inclined, do seem to converge towards a distant vanishing point just as other lines do, it is not necessary to represent this convergence, since their perspective representations in the plane of the picture also seem to converge as they recede from the eye, and in the same degree, covering and coinciding with them. The perspectives of the lines are themselves foreshortened, and the space between them diminished by distance.

27. To obtain this effect, however, in due degree, as, indeed, to obtain the just value of all other perspective effects, the eye of the spectator must remain at the station point. From other points the picture necessarily

looks inexact or distorted. These distortions increase from the centre outward; and since it is so inconvenient as to be practically impossible to keep the eye always at the station point, it is best, in order to keep this distortion within reasonable limits, not to extend the picture more than 60°, *i. e.*, not to make it much wider than its distance from the eye.

Some other phenomena relating to perspective drawings are represented in Plate I., Figure 2.

In this plate, though a variety of objects are indicated, only one direction of each kind is employed. All the right-hand horizontal lines belong to a single system, and all the left-hand lines to another. This, of course, would not happen to this extent in nature; but we have imagined, for simplicity's sake, that in the scene represented all the buildings are parallel, and that all the roofs are of the same pitch.

Plate I.

The Centre of the picture, V^C, the point nearest the eye and opposite the station point, S, is here not exactly in the middle of the picture, but considerably to the right, being just below the church on the hill. The Station Point is about six inches from the paper; and the eye must of course occupy this position in order to make the things represented appear of their proper shape and size.

28. In this plate the following notation is adopted, a notation that will be adhered to throughout this treatise, and with which the reader should make himself perfectly familiar. Each direction is indicated by a

Notation.

single letter, the direction of each system of planes by two letters, which give the direction of two elements of the system; each vanishing point by the letter V, with the letter denoting the direction of the lines to which it belongs written after it; and the horizon of each set of planes by the letter H, and the letters denoting the plane.

LINES AND THEIR VANISHING POINTS.

Lines.	*Their Direction.*		*Their Vanishing Points.*
C.	Horizontal and normal.	V^{C}	Vanishing point of normal lines. "the Centre."
Z.	Vertical. (To the zenith, or nadir.)	V^{Z}	" " vertical lines.
R	Right-hand horizontal [*i.e.*, horizontal lines going off to the right.]	V^{R}	" " Right hor. lines.
L.	Left-hand horizontal.	V^{L}	" " Left " "
M.	Right-hand inclined upwards.	V^{M}	" " Right incl. "
M′.	" " " downwards.	$V^{M'}$	" " " " "
N.	Left " " upwards.	V^{N}	" " Left incl. "
N′.	" " " downwards.	$V^{N'}$	" " " " "
P, Q / P′, Q′	Inclined lines formed by the intersection of inclined planes.	V^{P}, V^{Q} / $V^{P'}$, $V^{Q'}$	" " / " " Lines of Intersection

The optical lines are lettered R^0, L^0, C^0, etc. C^0 is the Axis.

If there are several lines having the same general inclination they may be distinguished by figures, as R^1, R^2, R^3, etc. Special vanishing points may be indicated as V^1, V^2, V^3, etc.

PLANES AND THEIR VANISHING TRACES (OR HORIZONS).

Planes.	*Their Direction.*		*Their Vanishing Horizon.*
RZ.	[Any plane of the system which contains, or is parallel to R and Z, *i.e.*, right-hand vertical planes.]	HRZ.	Horizon of the right-hand vertical planes.
LZ.	To L and Z, *i.e.*, left-hand vertical planes.	HLZ.	Horizon of the left-hand vertical planes.
RL.	To R and L, *i e.*, horizontal planes.	HRL.	Horizon of horizontal planes (*i.e.*, THE HORIZON).
RN.	To R and N, "inclined up to the left.	HRN.	Horizon of the planes RN.
RN′.	To R and N′ " " down " "	HRN′.	" " " RN′.
LM.	To L and M," " up " right.	HLM.	" " " LM.
LM′	To L and M′," " down " "	HLM′.	" " " LM′.

29. The position of the various vanishing points, as well as the dimensions of the various objects, are supposed to be obtained, in this picture, as they would be obtained in a sketch from nature, or from the imagination.

To find the Vanishing Point of a system of lines.

If we place the eye at the station point and *look* in the direction followed by a system of lines, we shall see their vanishing point (8); and if the plane of the picture is interposed we shall see the perspective of the vanishing point in the same direction, covering the real vanishing point. Hence the perspective of the vanishing point of any system of lines is found by passing through the station point an element of that system. The point where it pierces the picture-plane is the perspective of the vanishing point of the system, coinciding with and covering the real vanishing point.

30. As M and M′ are equally inclined to the horizontal plane, one looks up towards V^M at exactly the same angle that he looks down towards $V^{M'}$. V^M is accordingly just as far above the Horizon as $V^{M'}$ is below it.

The same is true of V^N and $V^{N'}$. But although M and N make the same angle with the ground, the distance of their vanishing points above the Horizon is not the same. For the eye at S, six inches in front of V^C, is further from V^L than from V^R, L being less inclined to the plane of the picture than R is. Looking up, then, at the same angle, though the eye sees the real left hand upper vanishing point at the same height as the right-hand one above the real Horizon, it sees its perspective, V^N, higher up on the paper.

31. In like manner the perspective of the horizon of a plane, or of a system of planes, is found by passing through the station point an element of the system. The line where it intersects the plane of the picture is the perspective of the horizon of the system of planes.

To find the Horizon of a system of planes.

For if the eye is at the Station Point, and glances along the element of the system passing through it, it will see the horizon upon the plane of the picture, covering and coinciding with the distant horizon. This line is the line where the plane passing through the eye cuts the plane of the picture.

32. And as the horizon of a system of planes passes through the vanishing points of all the lines that lie in it or are parallel to it, so does the perspective of the horizon pass through the perspective of their vanishing points; and as all lines lying in or parallel to the system have their vanishing points in this horizon, so do the perspectives of all such lines have their point of convergence or vanishing point in its perspective.

33. The propositions I., II., and III. (13), are thus as true for the perspectives in the plane of the picture as for the real lines and lines, vanishing points and horizons, as is exemplified over and over again in this plate. All the horizons shown pass through several vanishing points, and every vanishing point lies in the horizon of some system of planes. Every line which lies in two planes, as most of these planes do, has its vanishing point in both horizons, that is, at their intersection; and

the horizons of all the planes parallel to any one of these lines intersect each other at its vanishing point.

34. It is specially to be noted that the lines of the hips and valleys lying at the intersection of two planes of the roofs have their vanishing points at the intersection of the horizons of these planes.

Lines of Intersection.

Thus the lines P, P′, Q, and Q′, being at the intersection, respectively, of R N and L M, R N′ and L M′, R N and L M′, and R N′ and L M, we have

V^P	at the	intersection	of H R M	and	H L M.
$V^{P'}$	"	"	H R N′	"	H L M′.
V^Q	"	"	H R N	"	H L M′.
$V^{Q'}$	"	"	H R N′	"	H L M.

V^Q, the vanishing point of Q, is off the paper, being at the intersection of the horizons H R N and H L M′; and so in like manner is that of Q′ at the intersection of H R N′ and H L M.

On most of the roofs the planes L M′ and R N′, being on the further side, are out of sight. But the roof in the extreme foreground shows all four slopes. It has accordingly been selected for lettering.

35. This plate shows also that if the picture is vertical the horizon of a vertical plane, such as L Z or R Z, is a vertical line. For it must pass through the vanishing point, V^Z, of the vertical lines that lie in it, and this point is the infinitely distant zenith.

Vertical planes.

Besides, it is the line in which that plane of the system which passes through the eye intersects the plane of the picture (31); and as both these planes are vertical, their intersection must be vertical.

36. Hence the hips and valleys, P and P′, which lie in parallel vertical planes, and accordingly have their vanishing points V^P and $V^{P'}$ in the horizon of the system to which those planes belong, lie in a vertical line, one exactly above the other. As P and P′ are equally inclined to the horizontal plane, V^P and $V^{P'}$ as well as V^M and $V^{M'}$, or V^N and $V^{N'}$, are equally distant from the Horizon. These relations are indeed sufficiently obvious from the symmetry of the figure.

The vertical horizon HPP′ is not shown in the plate.

37. The proposition that all lines lying in a plane have their vanishing points somewhere in the horizon of that plane, receives special illustration in the case of the paths which cross the flat open space beyond the railroad; being level, they have their vanishing points on the horizon, one at V^R, others at V_3 and V. The ladder lying on the roof to the right has the vanishing point of its sides, which are supposed to be made parallel, at V^1 in the horizon of the plane of the roof, HRN′.

This proposition is very serviceable in putting in any parallel lines on any plane, as for instance, in drawing the diagonal lines of slating on the roof to the left, the vanishing points being shown at V_4 and V_5 in the horizon of RN, the plane of the roof.

38. But if a line lying in a given plane is parallel to the plane of the picture, then its vanishing point, though still in the horizon of the plane, will be at an infinite distance; and the line itself, being still directed to its vanishing

Lines parallel to the picture parallel to the horizon of the planes they lie in.

point, will be parallel to the horizon. For lines which meet only at an infinite distance are parallel. Hence we have the general proposition that, in any plane, a line which is parallel to the picture will be drawn parallel to the horizon of the plane.

30. In the figure the broken line *p p p* is drawn parallel to the plane of the picture. It is the line of intersection of a vertical plane parallel to the picture, cutting across the front corner of the principal building. In each plane it is parallel to the horizon of that plane.

Sections taken parallel to the picture.

40. That this must be so follows not only from 38, but also from the general proposition, that, if one system of parallel planes intersects another system, their lines of intersection are all parallel.

For a line lying in any plane, and parallel to the plane of the picture, may be regarded as the intersection of that plane by a plane parallel to the picture. But the horizon of the system of planes in which the line lies is the line in which a plane parallel to those planes and passing through the eye intersects the plane of the picture. We have thus two inclined planes parallel to each other, intersecting two vertical planes parallel to each other. Their intersections are accordingly parallel, and the line in question is parallel to the horizon of the inclined plane in which it lies; and since it is parallel to the picture, its perspective is parallel to itself, and also is parallel to the horizon: *Q. E. D.*

The intersection of any plane with the plane of the picture.

41. Moreover, if any plane of that system of planes is extended so as to cut the plane of

the picture, that intersection is also parallel to the others, and to the horizon in question.

42. This plate illustrates also the proposition (12) that in the case of solid objects the plane surfaces by which they are bounded are visible only on the sides turned towards their horizons (12); they are visible only when they are, so to speak, *below* their horizons. We see in the plate that the roof most nearly below the eye, being below all the horizons, shows all its slopes, and so does the next one to the left. In all the others, one, two, or three planes disappear, as they are above their horizons; until, at last, in the case of the church on the top of the hill, which is above all the traces, all the roofs are out of sight. One of the houses in the fort on the lower hill shows the roof LM just disappearing, the lines L and M both coinciding with the horizon HLM.

Planes visible only when below their horizons.

It is to be observed that these horizons, being portions of great circles, do not terminate at the vanishing points by which their position is determined, but pass through and beyond them.

43. The perspective of a line or of a point is often called a perspective line or point, or when speaking of the picture, simply a line or point. But in this last case, to avoid confusion of mind, one must be careful to notice whether the real line or point and the infinitely distant vanishing point are spoken of, or their representations in the plane of the picture.

So when one is speaking of the horizon of a system

of planes, he must be careful to notice whether the infinitely distant line, or the line in the picture which covers, and to the eye coincides with it, is intended.

CHAPTER III.

SKETCHING IN PERSPECTIVE. THE PERSPECTIVE PLAN. THE DIVISION OF LINES BY DIAGONALS.

HAVING in the first chapter considered the nature of the phenomena with which perspective drawings have to do, we examined in the last chapter the aspect of the drawings themselves, first observing the relation which lines parallel to the plane of the picture bear to their perspective representations, and then, in the case of those not parallel to the plane of the picture, the relation that the perspective lines and planes, by which the objects represented are defined, bear to the perspective of their vanishing points and horizons.

Plate II. illustrates almost all the points raised in explaining Plate I.; the roofs that are below their horizons being all visible, and those that are above them being all out of sight, while all the lines of intersection of the planes converge to the intersection of their horizons. This is specially noticeable of the *valleys* of the main roof in the lower picture. Plate II.

The horizon H P P′ of the system of vertical planes P P to which these valleys are parallel, and which accordingly passes through their vanishing points (13, II.), and which was not drawn in the previous plate, is here shown.

Since the lines which indicate the position of these valleys in the perspective plan lie in the same vertical planes with the valleys themselves, they must have their vanishing point in this same vertical horizon H P P′; and since, like all the lines of this plan, they lie in a horizontal plane, their vanishing point must lie in the Horizon (13, I.): it is therefore to be sought at the intersection of H P P′ with the Horizon, at the point marked V^X (13, III.). Since the vertical planes in which these valleys lie are obviously at an angle of 45° with the principal vertical planes R Z and L Z, this line X, whose vanishing point is at V^X, is at 45° with the lines R and L. V^R is off the paper.

The Vanishing Point of 45°.

44. If we put the eye at the station point S, four or five inches in front of V^C, and, looking first at V^R and V^L in directions at right angles to each other, look then between them so as exactly to divide the angle, we shall be looking in the direction X, and shall see V^X directly in front of the eye (6).

We will retain this notation, V^X, throughout this treatise, to denote the vanishing point of horizontal lines making an angle of 45° with the principal horizontal lines, R and L; and shall call it, for brevity, the vanishing point of 45°.

The little building on the left has steeper roofs than the other, their slope being the same as that of the roof of the tower. Their vanishing points are accordingly V^{N1}, V^{M1}, $V^{N1\prime}$, etc., which are off the paper.

As the tower roof is supposed to slope alike all round, the hips P_1 and P_1' lie also in parallel planes, at 45°;

their projection on the perspective plan has V^x for its vanishing point, and V^{pl} and $V^{pl'}$ lie in the horizon of the plane P P′, at equal distances above and below the Horizon (36).

45. The position of these vanishing points and horizons is supposed to be determined just as the position of the other leading points in the picture is determined; that is to say, their relative position on the paper is made to correspond to the relative position of the real points and vanishing points, as nearly as may be, by the eye, by looking first at the point, and then looking for the corresponding place on the paper. The position of the leading vanishing points being thus determined, the horizons can now be drawn connecting them, and new vanishing points, such as V^p and $V^{p'}$, determined by their intersection.

If the propositions illustrated in the last chapter are borne in mind, a consistent and tolerably correct perspective sketch can easily be made, the eye being greatly aided in its estimate of the relations of things, and their apparent shape and dimensions, by the considerations to which attention is thus directed. The principal points being fixed by the eye, the other points are then determined, partly by the eye, partly by means of lines drawn to the vanishing points.

Sketching.

46. A great advantage may also be found in the use of a *perspective plan* of any object that is to be drawn, especially in sketching, not from ob-

The Perspective Plan.

jects but from the imagination. Thus in the figure
Figure 5. (Fig. 5), although the main building could be drawn without much chance of error, it is by no means so easy to determine just where the tower behind it should make its appearance over the roof. By completing the plan, however, as is done by dotted lines, its position is at once determined. The objection that it is undesirable to cover the drawing with construction lines may be entirely met by drawing the plan at a lower level, as if it were the plan of the bottom of the cellar, ten or twenty feet underground; and for the purpose in hand, the cellar may be supposed to be of any convenient depth, so as to get the plan entirely out of the picture, as is done in the figure.

47. This sinking of the perspective plan has two incidental advantages. In the first place, it makes it practicable to draw it on a separate piece of paper, which may be removed and kept for use a second time, if, as often happens, a perspective drawing needs to be made over again. In the second place, it defines the positions of things much more accurately; the lines by whose intersection the position of the vertical lines is determined cutting each other more nearly at right angles. It will be seen in the figure that the lines in the real floor-plan cut each other so obliquely that it is not easy to tell exactly where the corners of the tower do come.

48. It follows from this that the level at which the
Bird's-eye Views. object is to be shown in perspective is quite independent of the level chosen for its plan. This also is illustrated in Plate II., the same plan

serving for three representations of the building, at different levels, — one nearly even with the eye, with a bird's-eye view below, and with what might be called a *toad's-eye* view of it above. The same vanishing points being employed in all three sketches, the phenomena pointed out in the previous chapter, of the appearance and disappearance of plane surfaces according as they come below or above their horizons, are here again illustrated.

49. In thus sketching in perspective, whether from nature — that is, from a real object — or from the imagination, it will be found much easier to determine vertical magnitudes than horizontal ones; that is to say, it is easy to determine the position of horizontal lines, but not their length; and the length of vertical lines, but not their position.

In the sketch, for example, the position of the vanishing points, and the position and height of the front corner of the building to be represented, being once assumed or determined, other heights, whether equal or different, can easily be determined by means of parallel lines drawn to the vanishing points. The height of an object having been assumed in one part of the picture, an object of the same height can be put in anywhere else by the employment of parallel lines.

But though it is thus easy to represent the three gable-ends in this sketch as being of the same height, it is not so obvious how to draw them so that they shall all seem equally wide.

The Division of Lines.

50. Moreover, the subdivision of the perspective of vertical lines, whether into equal parts or according to some given proportion, presents no difficulty; for the vertical lines are parallel to the picture, and their perspectives will accordingly be divided just as the lines themselves are (20).

But while the division of vertical lines and their apparent diminution in size is easily managed, the subdivision of horizontal and inclined lines (except those which like the vertical lines are parallel to the plane of the picture) is a matter of difficulty. The more remote divisions are smaller, but it is not clear how much smaller.

Two methods are adopted to determine this, — the method of Diagonals, and the method of Triangles. Let us take the first, first.

The Method of Diagonals.

The method of Diagonals is illustrated in the various figures of Plate II. It applies to parallelograms whose perspectives are given or assumed the following propositions: —

Halving, quartering, etc.

51. *Proposition* 1. A line drawn through the intersection of the diagonals of a parallelogram, parallel to two of its sides, bisects the other sides and the parallelogram itself.

Fig. 3, *a*.

This process may be repeated with each half, and the given figure, or any line in it, divided into 2, 4, 8, 16, or 32 equal parts, etc. See Fig. 3, *a*.

Fig. 5.

The application of this to the perspective of a parallelogram is shown in Fig. 5, where the left-hand side of the larger building is thus divided.

52. Less familiar is the employment of this principle to ascertain the vertical axis of a tower two of whose sides are given in perspective, as in Fig. 5. If diagonals are drawn across the tower, from two points on the right-hand vertical corner to points at the same levels on the left-hand corner, they will intersect in the middle of the tower, and a vertical line through their intersection may be used to determine the apex of the roof which covers it, as in the figure. These diagonals lie in a vertical plane that crosses the tower diagonally.

53. This is the common way of dividing a perspective line or surface into halves; and it is constantly used, as on the left-hand side of this building, and on the right-hand side of the building above (Fig. 4), to determine the centre line of a gable, and the position of its apex. Fig. 4.

54. It is obvious that this furnishes an alternative method of determining the slope of these roofs. Instead, that is, of fixing the position of the vanishing points of M and M′, P and P′, and thus obtaining the direction of these inclined lines, we may assume at once the direction of any one of these lines, say the nearest one. The intersection of this line with the central vertical line fixes the height of the roof; the other slope and the other roofs are then easily drawn.

55. Perspective is full of these alternative methods, different ways of doing the same thing. Which way it is best to adopt in any given case, depends upon the nature of the case. In the present instance, the vanishing points V^M and $V^{M'}$ being outside the picture, the method of diagonals is rather the most convenient.

56. It is to be observed, however, that though V^{M} and V^{M1} are off the paper, V^{P} and V^{P1} are within easy reach. It is generally worth while, accordingly, to fix the position of the more remote vanishing points, so as to determine the position of the traces or horizons that lie between them, and of the points where those horizons intersect, even if we make no direct use of the vanishing points themselves. Thus, in the plate, although the points V^{M1}, V^{N1}, $V^{M1'}$, and $V^{N1'}$, which give the slope of the roofs of the small house and of the tower, are all at a distance, the horizons of the planes of the roofs $H R N_1$, $H L M_1$, $H R N_1'$, and $H L M_1'$ all cross the paper, and their intersections V^{P1} and $V^{P1'}$ are close at hand.

57. *Proposition* 2. If through the intersection of the diagonals a second line is drawn parallel to the other two sides of the parallelogram, a single diagonal suffices to effect the subsequent subdivisions, as is exemplified in

Fig. 3, *b*. Fig. 3, *b*, and on the left-hand side of the larger

Fig. 6. building in Fig. 6 below.

58. *Proposition* 3. *Conversely*, if a line drawn from

Doubling, tripling, etc. one corner of a parallelogram to the middle of one of the opposite sides be continued until it meets the other side, prolonged, the length of that side, or of the parallelogram itself, may be doubled, and by

Fig. 3, *c*. a repetition of the process, tripled, quadrupled, etc. See Fig. 3, *c*.

This proposition is of great use in perspective drawing, as may be seen in Fig. 5, where the gabled end on the right is several times repeated, each time smaller than before.

It will be seen that the gable ends of the roofs grow steeper and steeper, their lines converging, in fact, to the distant vanishing points M and M′. By obtaining those points, the accuracy of these results can be tested.

59. *Proposition* 4. If one side of a parallelogram be divided in any way at one end, equal divisions may be laid off at the other end by means of two diagonals. See Fig. 3, *d*.

Symmetrical division.
Fig. 3, *d*.

This is very useful in giving a symmetrical treatment to a surface shown in perspective, as is seen in the left-hand building, Fig. 5. The position and width of the nearer window on the side of the building being assumed, the vertical lines enclosing the further window are easily found.

At the end of the building the inclined lines of the gable, which may be regarded as the semi-diagonals of an unfinished parallelogram, answer the same purpose. The base of any isosceles triangle can be divided in this way.

60. *Proposition* 5. If one side of a parallelogram be divided in any way, the adjacent sides may be similarly divided into proportional parts, by means of one diagonal; and by using the other diagonals the order of the parts may be reversed. See Fig. 3, *e*.

Proportional division.
Fig. 3, *e*.

By this means any required division of a line given in perspective may be effected, as is shown in Fig.

6, on the right-hand sides of both buildings. The required division is made on the vertical line, and then transferred to the horizontal line by means of the diagonal, the nearest corner of the small house being divided according to the desired position of the door and windows, and that of the large building into three equal parts.

61. If the diagonal makes an angle of 45° with the adjacent sides, their segments will of course be not only proportional, but equal, each to each.

In the perspective plan of the small building, for example, in which the diagonals are directed to V^x, the "vanishing point of 45°," and accordingly make an angle of 45° with the sides of the building, it appears that the window is just as far from the corner on one side as the farther edge of the door is on the other. It appears also that the plan of this building is just four squares, though it hardly looks so, the side being greatly foreshortened, while the main part of the other building is just as broad as it is long, comprising nine squares, each as large in plan as the tower.

62. In applying this proposition to a perspective drawing, the line on which these parts are first laid off must of course be a vertical line, or some other line parallel to the plane of the picture, as it is only in the case of such lines that the division of the perspectives is proportional to that of the lines themselves (20).

63. *Proposition* 6. It is not necessary that the length of this line shall be previously determined. Indeed,

it is more convenient that it should not be, as it is easier to establish a given ratio of parts on an indefinite line. The equal or proportionate parts may be set off at any convenient scale, on any convenient line that touches the end of the line to be divided, and the diagonal drawn without completing the parallelogram, as in Fig. 3, *f*. Fig. 3, *f*.

The division of the long wall in Fig. 6, for instance, is effected by setting off three equal distances upon the further corner, just as well as by dividing the near corner into three equal parts.

64. It is not necessary in any of these cases, of course, that the parallelogram shall be a rectangle. The inclined line N, for example, in the middle of the upper figure, Fig. 4, is divided into four equal parts by equal divisions laid off on the vertical line that bisects the gable.

In these last propositions, it will be observed, use has been made of only half a parallelogram, that is to say, of a triangle.

65. Proposition 5 may then be restated as follows : —

If one side of a triangle be divided in any way, the adjacent side may be divided into proportional parts by means of lines drawn parallel to these two sides and meeting on the third side. See *f*, Fig. 3. Fig. 3, *f*.

66. And from Proposition 6 we may derive this : —

If from one end of a line there be set up an auxiliary, parallel to the plane of the picture, any parts taken upon

the perspective of this auxiliary may be transferred to the perspective of the line, in their true proportions, by means of a third line joining the last point taken on the auxiliary with the other end of the first line (65).

Random lines used for proportional division.

67. But as any line, drawn in the picture at random, may be conceived of as being the perspective of a line, which it exactly covers and conceals, drawn parallel to the picture, it follows that *any line whatever, touching one end of a perspective line, may be used as an auxiliary by which to divide it in any required proportion*; and, the triangle being completed, the first segments of the broken lines by which the proportions are transferred will be parallel to the line to be divided and will be directed to its vanishing point, and the second segments will be actually parallel to the auxiliary line, since its vanishing point is at an infinite distance.

Thus, if it is required to erect six equidistant spikes upon the ridge of the left-hand building, we may from either end of the ridge draw a line in any convenient direction, and lay off on that line five equal parts, using any convenient scale, as in the figure. Completing the triangle and proceeding as above, we get the points of division desired. This triangle does not lie in a vertical plane but in an oblique plane, containing both the horizontal line to be divided and the auxiliary line; this line is parallel to the picture, and is shown in its true direction. Fig. 5, Fig. 6.

Finally, since any line whatever that touches the end of the line to be divided will serve this purpose, it is

often convenient, instead of drawing a new line, to employ a line already existing. We may, for example, as in Fig. 6, lay off our five equal parts on the sloping line of the further gable, and obtain the same points on the ridge as before. This line, however, is now conceived to represent, not the line of the gable, but a line parallel to the plane of the picture, so taken as exactly to cover the line of the gable, so that their perspectives coincide.

CHAPTER IV.

THE DIVISION OF LINES BY THE METHOD OF TRIANGLES.

THE third chapter first set forth the convenience, in making a perspective drawing, of putting into perspective the plan of the object to be drawn, and of sinking this plan so far below the representation of the object as to get it quite free from the picture. Plate III., Fig. 10, affords further illustration of the use of the perspective plan. The plan of the gatehouse on the left is indeed below the picture, having in fact been drawn on another piece of paper, and removed, as suggested in a previous paragraph (47). But the plan of the one in the distance on the right is given, and it serves to determine all the principal horizontal dimensions. The plan of the principal building, the barn in the valley, is drawn above it, instead of below, as is sometimes most convenient, especially in high buildings, in the upper parts of which it is of advantage to have the perspective plan near at hand. It is often a convenience, also, to make several plans, set one above another, taken at different levels. In the plate, for example, we have first the plan of the outline of the walls, to determine the position of the doors and windows, and of the posts of the shed, and then just above it the plan of the

Plate III., Fig. 10.

eaves, showing their projection, and the position of the brackets beneath them. As only the front part of the building is seen, only the front part of the plan needs to be drawn. In putting in the eaves, advantage is taken of the "vanishing point of 45°," V^X, to make them equally wide on each side.

In this plate the slope of the roofs and gables of this building, as well as of the smaller one with a hipped roof beyond it, is indicated by the same letters as in the previous plates, and their vanishing points accordingly by V^M, $V^{M'}$, etc., as before. The gables of the little gate-houses are so steep that their vanishing points are quite out of reach; and these gables are, in fact, drawn by the method of diagonals, as described in the previous chapter (54). The slope of the steps is given by V^{M1} and $V^{M1'}$, and the trace of the inclined planes of the bank by $H L M_1$ and $H L M_1'$. Their position shows that the banks are a little steeper than the roof of the barn. The diagonal braces of the fence have nearly the same slope as the barn shed, converging to points just below V^N and just above $V^{N'}$.

In this plate the centre, $V^C_,$ is again quite out of the middle of the picture. The station-point, S, the proper position of the eye, is about six inches in front of $V^C_.$

The Method of Diagonals.

68. The first of the two methods by which a line given in perspective may be divided up in any given proportion has already been described. It was shown that this, though called the Method of Diagonals, finally leads to the division of

such a line by means of a triangle, one side of which is formed by the line to be divided, and one side by an auxiliary line, drawn parallel to the plane of the picture in any convenient direction and divided in the given proportion. The points of division are transferred from this auxiliary line first to the third side of the triangle, by lines parallel to the perspective line, and directed to its vanishing point; and then to the perspective line by lines actually parallel to the auxiliary.

69. Both these steps are obvious and simple applications of the proposition that lines drawn parallel to one side of a triangle divide the other two sides proportionally. But it does not yet appear what is the real direction of this third side, nor in what plane the triangle really lies; that is to say, the vanishing point of this line and the horizon of this plane are not yet determined.

The Method of Triangles.

70. The other method of dividing perspective lines, called, *par excellence*, the Method of Triangles, is a more direct application of the same principle. The auxiliary line, as before, is drawn parallel to the plane of the picture; but the points by which it is divided are now transferred directly to the perspective line by lines drawn parallel to the third side of the triangle. Plate III. is devoted to the illustration of this method.

Fig. 7, *a*, *b*.

Fig. 7, *a* and *b*, shows the difference between this method and the preceding. In each of the triangles here shown, the base is divided proportionally to the parts set off on the left-hand side. But in the upper ones the division is effected by the Method of Diagonals, as in Fig. 3, *f*; in the lower ones the

same result is reached, more directly and simply, by the Method of Triangles.

71. This application of the principle in question, however, though more direct and simple, is in one respect less easy of adaptation to lines given in perspective. For the two systems of parallel lines employed in the Method of Diagonals may be drawn without difficulty, the first having the same vanishing point as the line to be divided, and the second being actually parallel to the auxiliary line, since that line is parallel to the picture. But in the Method of Triangles, the lines by which the points are transferred are parallel to the third side of the triangle, whose vanishing point is not known. It is accordingly necessary first to find the vanishing point of this line.

72. This may be done at once, when, as in the plan of the eaves, at the top of the plate, the plane in which the auxiliary triangle lies is known; that is to say, when its horizon has been already ascertained. The auxiliary line here lies in the horizontal plane, and the given line lies in the same plane; the whole triangle is accordingly in the horizontal plane, and all its lines have their vanishing points in the Horizon, — the given line at V^L, the auxiliary line at an infinite distance, and the third side of the triangle at V_1. This point is ascertained simply by prolonging this side until it reaches the Horizon (13 I.). If now it is desired to find the position of the ten brackets that support the eaves, it is easy to lay off on the auxiliary line nine equal divisions, and to complete the triangle:

Point and Line of Proportional Measures.

Horizontal lines.

by drawing lines parallel to the third side, the distances set off on the auxiliary are at once transferred to the perspective line, the lines converging to V_1. This auxiliary vanishing point is called the vanishing point of proportional measures, or simply the point of measures. The auxiliary line is called the line of proportional measures, or simply the line of measures. These must not be confounded with the point and line of *equal* measures described in the next chapter (98, 99, 125).

73. It makes no difference, of course, at which end of the perspective line the line of proportional measures is drawn, so that it is parallel to the picture. The relative position of the doors, windows, etc., in the lower plan, and of the posts of the shed, are laid off on lines of measures drawn from their further ends, and the points of division transferred to the lines of the plan by means of the points of measures V_2 and V_3, both of which, of course, are also on the Horizon. But it is obviously conducive to precision to have the line of measures touch the nearer end of the line to be divided, since, in general, converging lines give more accurate results than do lines of divergence.

Neither, of course, does the size of the proportional parts laid off upon the line of measures affect the result.

Fig. 7, *c*. In Fig. 7, *c*, the base of the triangle is divided into the same four equal parts, whether the parts taken on the adjacent side are large or small. Any convenient scale may be used; but that scale will in general be found most convenient which makes the line of measures about as long as the perspective line to

be divided, and which brings the point of measures within easy reach.

74. In the same way a line lying in a vertical plane may be divided by means of a vertical line of measures; the point of measures or vanishing point of the third side of the triangle and of the lines drawn parallel to it being now in the horizon of the vertical plane. If a line lies at the intersection of two planes, it is a mere matter of convenience whether the line of measures is taken in one plane or the other, or in which horizon the point of measures is taken.

Vertical lines.

Thus the seven parts into which the length of the barn in Plate III. is divided may be taken either on a horizontal or on a vertical line; that is to say, upon a line of measures parallel to the horizon of either plane. The points at the bottom of the wall, on the left-hand side, which determine the position of the doors and windows, may be got either by means of a horizontal line of measures, as shown, with its point of measures on the Horizon, at V_4, or by a vertical line of measures, namely, the corner of the barn, on which the same proportional parts are laid off, with its point of measures on the horizon of the plane L Z at V_5. Here the first triangle lies in the horizontal plane, and the second in the vertical plane, the first on the ground and the second in the side of the barn, as they seem to.

75. If a line lies in a plane inclined to the horizontal plane, as each inclined line of the gable-ends of the barn lies in the plane of its roof, a simi-

Inclined lines.

lar procedure may be followed. A line of measures may be taken in that plane, touching the given line at one end and parallel to the picture, the point of measures being now in the horizon of the plane of the roof.

76. And as in the horizontal plane a line parallel to the picture is horizontal, and in vertical planes vertical, — that is to say, in each case parallel to the horizon of the plane it lies in, — so in the case of an inclined plane, a line lying in it parallel to the picture is parallel to the horizon of the system to which the plane belongs; for though its vanishing point is somewhere in that horizon (13, I.), it is at an infinite distance (20). A line directed to that point is accordingly parallel to the horizon.

77. The perspective of an inclined line can then be divided in any required proportion, as easily as that of a horizontal or vertical one, by drawing through one end of it a line of proportional measures parallel to the horizon of the inclined plane in which it lies, and taking the point of measures on that horizon.

Thus in the plate the position of the brackets or purlins on the gable of the barn is found by dividing each slope into six parts, by means of a line of measures drawn parallel to H L M, the horizon of the roof in question; and as the sloping lines of the gable lie in a vertical plane R Z, parallel to the side of the barn, as well as in the plane of the roof, it follows that the position of the six brackets can be found either by laying off equal parts, on vertical lines, with points of measures on the horizon of R Z, at V_6 and V_7, or by laying off equal parts upon lines of measures parallel to the horizons of the planes of the

roofs, that is to say, parallel to H L M for the left-hand slope, and to H L M′ for the other, with points of proportional measures at V_8 in H L M, and V_9 in H L M′, respectively.

In the former case the triangles lie in the plane of the gable-end; in the latter, each lies in the plane of its own roof.

78. It follows from the above, as has already been shown, that if any object bounded by plane surfaces be cut through by a plane parallel to the plane of the picture, the line of intersection on each face will be parallel to the horizon of the plane in which it lies (39). This is exemplified in the plate, where the dotted line, A A, running along the ground and over the barn, follows this law. If the front corner of the building were sliced off parallel with the picture, this would be the line of the cut. The same thing is exemplified on the front corner of the other building.

Sections parallel to the picture.

We shall find use for this by and bye, when we come to the perspective of shadows.

79. Finally, just as in the Method of Diagonals we found at last that the auxiliary line, or line of measures, may be taken in any direction, at random, so here the same thing is true. For here too any line, drawn in any direction at random, from either end of a perspective line, may be regarded as the perspective of a line of measures, parallel to the picture and drawn parallel to the horizon of the plane in which it lies. This horizon, then, will be parallel to it; and

Random lines of proportional measures.

since the plane contains the perspective line, its horizon must pass through the vanishing point of that line; for the horizon of a plane passes through the vanishing points of all the lines that lie in it (13 *c*); if then through the vanishing point of the line we wish to divide, we draw a line parallel to the assumed line of measures, we shall have the horizon of a plane in which they both lie; and upon this horizon the third line of the triangle, joining the other end of the perspective line with the last point taken on the line of proportional measures, will have its vanishing point. This point, the point of measures, can be found, just as before, by prolonging the third line, the base of the triangle, till it touches it.

80. The principle that the line of measures may be drawn at random in any direction, the corresponding point of measures being taken on a line or horizon drawn parallel to it through the vanishing point of the line to be divided, is illustrated in the division into five equal parts of the hip of the roof of the smaller building in the middle distance. Here the line of measures is drawn arbitrarily at about 60°, the auxiliary horizon being drawn through V^p, the vanishing point of the hip, and its point of measures, V_{10} determined on that horizon.

The triangle here seems to lie in the plane of the roof, but in fact it has nothing to do with it.

81. Moreover, since the only characteristic of this auxiliary horizon, relatively to the conditions of the problem, is this, that it passes through the vanishing

point of the line to be divided, it follows that any line drawn through the vanishing point of a given line may be regarded as the horizon of a plane in which the given line lies, and will contain the point of measures corresponding to a line of measures drawn through either end of the given line parallel to it.

82. This gives us, in other words, this singular proposition: —

Perspective lines used as auxiliary horizons.

Of any two perspective lines having the same vanishing point, one may be taken as the horizon of a plane passing through the other; and if a third line be drawn parallel to the first, and touching one end of the second, any parts taken upon this third line may be transferred to the second in their true proportions by means of a point of measures taken upon the first. See Gwilt's Encyclopædia of Architecture, § 2457.

83. The position of the vertical bars of the cresting upon the ridge of the gate-house on the left is determined in this way, five equal parts being laid off upon a line drawn from the further end of the ridge parallel to the eaves of the roof, as a line of measures, and the point of measures, V_{11}, taken on the eaves.

The way in which the position of the vertical bars of the gate below is determined also illustrates this proposition. A line touching the top of the gate is drawn parallel to the ridge-pole, which has the same vanishing point, V^L. Equidistant points are taken on this line, and transferred to the top of the gate by a point of measures, V_{12}, taken on the ridge-pole. This reduces the

labor of dividing up a given perspective line in any required proportion to almost nothing.

84. Here, as in the corresponding case in the previous chapter, care is to be taken not to fancy that the line of measures, and the triangle determined by it, really lie in the plane they seem to lie in.

In this last case the triangle lies in an imaginary inclined plane, and is no more vertical, as it seems to be, than the point of measures is on the ridge, as it seems to be: it is really in the infinitely distant horizon which the ridge covers and coincides with.

Plate III. also furnishes illustrations of two points of general interest.

85. The first of these is the use of the point V^X, the vanishing point of horizontal lines making an angle of 45° with the principal directions R and L, to determine V^N, when V^M is given, the lines that slope up to the left being supposed to make the same angle with the ground as those that slope up to the right. If these inclinations are equal, the inclination of the planes R N and L M will be equal, as in the case of these roofs; their lines of intersection, P, will lie in vertical planes, making 45° with the principal vertical planes; their horizon will be a vertical line passing through V^X, as shown in the previous chapter (42); and V^P will be at the intersection of this horizon with H L M. If now H R N be drawn through V^R and N^P, V^N will be found at its intersection with H L Z, and $V^{N'}$ will be at an equal distance below.

The point V^{P1}, which determines the direction of the

line P_1, at the intersection of the two banks in the further corner of the barnyard, is found in like manner.

86. The second point is illustrated by Fig. 8, which shows how the true direction of the lines Q or Q′, whose vanishing points are at the distant intersection of the nearly parallel horizons H R N and H L M′, or H R N′ and H L M, may be obtained by means of two similar triangles, the common device for directing a third line to the intersection of two given lines, as shown in Fig. 9. Fig. 8. Fig. 9.

This is applied in the plate, Fig. 10, to find the true direction of the left-hand line of the hipped roof, just below the point V^C. Fig. 10.

CHAPTER V.

ON THE EXACT DETERMINATION OF THE DIRECTION AND MAGNITUDE OF PERSPECTIVE LINES.

THE first two chapters were given to a general observation of the phenomena of perspective, in nature and in drawings, and the last two to an explanation of the practical making of such drawings, certain data being assumed. It was assumed that the position of the principal vanishing points, giving the direction of the principal lines, had been already determined, with more or less accuracy, by the eye or by the judgment, and that their length had also been fixed in the same way. The discussion showed how the position of other vanishing points and the length and direction of other lines could then be determined, and how any of the lines thus drawn could be divided up in any desired manner, that is to say, in any given proportion.

It is necessary, in order to conclude this part of the subject, to show how these data may be more precisely determined. It is necessary to show how, when the real direction of lines is exactly known, their vanishing points may be fixed with precision, and how, when their

real length is known, the exact length and position of their perspective representations may be determined. The position of the object to be drawn, the position of the picture, and the position of the spectator's eye, must, of course, be known also.

Plate IV. shows how these questions are answered: Fig. 11 showing in plan, and upon a reduced scale, the position of the spectator at the station point S; that of the picture at *p p*, which shows the plane of the picture edgewise as it would appear looking down upon it; and that of the object to be represented at A. Two elevations of this object, which is a small house, showing its vertical dimensions, are given alongside; the plan and the two elevations together giving exact information as to the magnitude and direction of the lines defining it. The picture itself is shown between the plan of the house and its own plan, just as if the plane of the picture, *p p*, had been revolved backward into the plane of the paper.

Plate IV. Fig. 11.

The object is here represented as being about six times as far from the station point as the picture is, the picture being about eighty feet from the station point, and the nearest corner of the house about thirteen feet. It is this relation that obviously determines the *scale* of its perspective representation, which would be greater if the picture were farther from the station point, or the object nearer, and *vice versa*. But we shall come to the question of scale presently (94).

87. The first question is that of the direction to be given to the various perspective lines; we must determine the vanishing points of these various systems of lines, horizontal and inclined. Vertical lines, and all other lines parallel to the picture, will of course have their perspectives parallel to themselves (20). The horizontal lines belong to three systems, the directions of which are indicated in the plan of the little house as R and L, going off to the right and to the left, at right angles to each other; and X, dividing the angle between them, and making an angle of 45° with each. If now the spectator, standing at S, looks in a direction parallel to R, he will see the vanishing point of that system of lines directly before his eye; that element of the system which passes through S is in fact seen endwise, appearing as a point covering and coinciding with the vanishing point of the system of right-hand horizontal lines, which is in the infinitely distant horizon (6). The perspective of this vanishing point, V^R, in the plane of the picture, will be found exactly where this element pierces the picture, that is, where it crosses the line *p p*. V^L and V^X can of course be found in the same way; and the centre of the picture, V^C, the point nearest the station point and at the other extremity of the axis $S\,V^c$, is easily determined at the same time. Since R and L are at right angles, the triangle $V^R S V^L$ is a right-angle triangle, and S lies on the circumference of a semicircle of which V^R and V^L give the diameter. In the picture itself, just above, these points of course appear on the Horizon; for since these lines are all horizontal,

The Directions of Horizontal Lines.

their vanishing points lie in the horizon of the horizontal system of planes (13, I.). H R Z and H L Z, the horizons of vertical planes parallel to the sides of the building, can now be drawn, as usual, through V^R and V^L; and H P P′, the horizon of the diagonal planes, through V^X. The vanishing points of any of the other horizontal lines, and the horizons of any other vertical planes could of course be found in a similar manner. A line perpendicular to the plane of the picture, and accordingly parallel to the Axis, would of course have its vanishing point at V^C.

88. It only remains to find the vanishing points of the inclined lines M and M′, N and N′, and thence as before, the horizons of the roofs, and the vanishing points, P and P′, of their hips and valleys. This is easily done by the aid of the elevations, which show the real inclination of these roofs and gables to be 60° for the lower slope, and 30° for the upper. If the spectator at S, then, while looking at V^R in the direction R, should raise his eyes at an angle of 30°, he would see V^M, the vanishing point of the upper slope of the gable, directly before him, the triangle $V^M V^R S$ being right-angled at V^R. If now this triangle were revolved about the vertical side $V^R V^M$, so as to bring the station point, S, into the plane of the picture at D^R, it would appear in the picture above as the triangle $V^M V^R D^R$, the angle at D^R being 30°.

The Directions of Inclined Lines.

Fig. 13 gives a perspective view of the plane of the picture *pp*, with the eye at the station point, S, in front of it; the triangle in question

Fig. 13.

is shown both in its original position and also as it appears when swung round into the plane of the picture.

89. It follows from this that if from V^R the distance $V^R S$ is laid off along the Horizon, we obtain the point D^R; and if from this point we draw a line at an angle of 30°, we shall obtain V^M at its intersection with H R Z. In the same way D^L and V^N may be obtained by setting off the distance of V^L S along the Horizon from V^L, and drawing the line $D^L V^N$, also at an angle of 30°.

Points of Distance.

The points D^R and D^L are called the right-hand and left-hand points of distance; they show the distance of the station point from the right-hand and left-hand vanishing points. It is to be observed that D^R is found on the left and D^L on the right of V^C.

In the same way the vanishing point of every other horizontal line, as, for example, of X, has its corresponding point of distance, found by setting off along the horizon its distance from S. Thus we have $V^X D^X$ equal to V^X S.

90. $V^{M\prime}$ and $V^{N\prime}$ will of course be seen as far below the Horizon as V^M and V^N are above it, and V^P and $V^{P\prime}$ will be at the intersection of the horizons of the inclined planes R N and L M, R N′ and L M′, as before.

Fig. 13 gives further illustration of most of these points. To prevent a confusion of lines, the centre, V^C, is taken on the left-hand side of the picture instead of on the right-hand side, as in Fig. 11.

91. By a reverse process, when the vanishing point of inclined lines is known, their real inclination can be discovered by drawing a line from this vanishing point to the point of distance of the horizontal line beneath them. Thus in the figure a line from V^P to D^X would give the angle $V^P D^X V^X$, which is the true slope of the line of intersection of the upper roofs.

To find the true inclination of inclined lines whose perspectives are given.

92. If the lines M and N have different inclinations, the point V^P will of course not come over V^X, and the distance must be measured from the point of the horizon that it does come over.

The vanishing points of the steeper slopes of the lower roofs are found in like manner at V^{M1}, V^{N1}, etc.

93. The exact direction of perspective lines being thus determined, since the position of their vanishing points is thus exactly fixed, it now only remains to determine their length, and the position of some one point in each. For such lines as are parallel to the picture this is easy, for every such line may be considered to lie in a plane parallel to the picture, the centre of which, or point in the plane nearest the eye, will have its perspective at V^C, the centre of the picture; the perspective of the line in question will be parallel to the line itself, and its length and its distance from the centre, V^C, and all other lengths and distances taken in that plane, will be less, as we have just seen (86), in proportion as the distance of the plane from the plane of the picture is greater. If, as in Fig. 11,

The length of lines parallel to the picture.

the plane *m m* is six times as far from the spectator at S as the picture *p p*, all lines in *m m* will be drawn at one sixth of their original size, and be at one sixth their distance from the centre. The front corner of the house, for instance, which lies in the plane *m m*, is so drawn.

Fig. 11.

The Plane of Measures.

This imaginary plane *m m*, which is generally drawn through the nearest part of any object to be represented, is called a *plane of measures*, and, like the picture, is defined in position by the length and position of its axis, which coincides with that of the picture, but is generally a great many times as long.

Its relative distance behind the plane of the picture is commonly, of course, much greater than that shown in the figure, in which, for perspicuity's sake, the picture is represented as being about eleven feet across and about thirteen feet from the spectator. The picture is commonly set only a few feet off, while the object represented is often a hundred times as many.

Sometimes, for convenience, two planes of measures are employed at different distances, lines in the further one being drawn to a smaller scale than in the nearer one.

Lines in the Plane of Measures.

94. It follows from what has been said that any line drawn in a plane of measures in any direction, horizontal, vertical, or inclined, is also parallel to the picture, and that its perspective will be parallel and proportional to it, but on a smaller scale. This scale depends on the relative distance of the picture plane and the plane of measures from the eye, or station point.

If the latter is twice or ten times as far away, lines drawn upon it will be presented in the picture one half or one tenth full size. *All lines in a plane of measures have their perspectives drawn to the same scale.*

Scale of the drawing.

It is common, in the case of large objects, such as buildings, to set the picture at just $\frac{1}{96}$, $\frac{1}{144}$, or $\frac{1}{192}$ of the distance of the plane of measures, *i. e.*, of the object. Lines in the plane of measures are then represented $\frac{1}{96}$, $\frac{1}{144}$, or $\frac{1}{192}$ full size, etc.; that is to say, on a scale of $\frac{1}{8}$, $\frac{1}{12}$, or $\frac{1}{16}$ of an inch to the foot.

The centre of the plane of measures coincides in perspective with the point V^C, the centre of the picture (93). The perspective of any other point in the plane of measures may be found by laying off its distance, according to the scale, in its real direction, in the plane of the picture.

Fig. 12.

95. Fig. 12, in which the various vanishing points and traces, and the points of distance D^R and D^L, are determined as in the previous figure, illustrates this practice. The picture is supposed to be about six inches from the eye, as in the case of our previous illustrations. This is less than is desirable, but is as much as the scale of these illustrations permits. The front corner of the building, through which the plane of actual measures is taken, is supposed to be a hundred and ninety-two times as far away, that is to say, ninety-six feet from the spectator, or ninety-five feet and a half behind the picture, the scale of the perspective of that corner being one sixteenth of an inch to

a foot. This corner, being eight feet high, is drawn half an inch high. All other lines in the plane of measures are drawn to the same scale, which is indeed the same scale as that to which the plan of the building is drawn in Fig. 11, and the elevations alongside. Dimensions can then be transferred directly from these drawings to Fig. 12, so long as the lines to which they apply lie in the plane of measures. Such a line is the line $v\ v$, the line in which the planes of the front and end walls of the house intersect the plane of measures. Any such vertical line, lying in the plane of measures, is called a line of vertical measures. By setting off upon this line the heights of any part of the building, by scale, they can be transferred to their proper places by horizontal lines directed to the vanishing points. The height of the gable in the figure is determined in this way. The line $m\ m$, also, in which the plane of measures intersects the horizontal plane on which the perspective plan is taken, is such a line. Its perspective, shown at $g\ l$, is called the Ground Line, or line of horizontal measures, and any dimensions can be laid off upon it at the same scale as upon the perspective of the vertical line; as it is parallel to the picture, the divisions of its perspective are proportional to those of the line itself.

Line of Vertical Measures.

Line of Horizontal Measures.

By prolonging the planes of the other walls until they intersect the plane of measures, additional lines of vertical measures are obtained, such as $v'\ v'$, in the plane of the further gable. In the same way every

horizontal plane gives a line of horizontal measures, as is shown in the case of the two perspective plans below.

96. For very small objects the plane of measures, and with it the object itself, is brought nearer, and may even coincide with the plane of the picture. In this case lines lying in it are drawn full size.

The plane of the picture the plane of measures.

Sometimes, instead of taking the object of its real size at its real distance, we suppose a miniature of the object to be set up near at hand, of any convenient scale. In this case also the object may be supposed to be close to the plane of the picture, and the plane of the picture to coincide with the plane of measures.

The model or miniature object.

This is illustrated in Fig. 11, in which a small plan of the house is drawn in contact with the plane of the picture, *p p*, just as a large plan, representing full size, is drawn in contact with the plane of measures *m m* above, the whole being drawn at a scale of sixteen feet to the inch. Or we may regard Fig. 11 as drawn on the same scale as Fig. 12, that is to say, full size, considering the plane *m m* to be the plane of the picture, six inches from the eye at S, with a miniature of the building, a model made to the scale of a sixteenth of an inch to the foot, just behind it. In this case Fig. 12 represents the picture drawn on the plane *m m*, just as the little picture in Fig. 11 represents that drawn on plane *p p*.

97. Having thus the means of drawing in any hori-

zontal or vertical plane a line, lying in a plane of measures, upon which dimensions can be laid off by scale, we have now to transfer these dimensions to other lines in the same plane.

The length of other lines parallel to the picture, by scale.

If these lines also are parallel to the picture and to the plane of measures, the case presents no difficulty. It is only necessary to draw parallel lines from one line to the other. In the figure, for example, the heights laid off on the line of vertical measures *v v*, at the front corner are transferred to the other corners and to other vertical lines by parallel lines directed to the vanishing points V^R and V^L. In this way the vertical dimensions of every part of an object, and the position of its horizontal lines, may be determined.

The length of any other lines parallel to the picture, horizontal, vertical, or inclined, may be obtained in a similar way from lines in the plane of measures parallel to them, and lying at the intersection of that plane with the planes in which they lie.

The length of horizontal lines inclined to the picture by scale.

98. To determine the length of the horizontal lines not parallel to the picture, and to lay off given dimensions upon the perspective of such lines, we can employ a method similar to the Method of Triangles described in the last chapter. By that method we laid off upon such lines parts *proportional* to parts taken upon a line of *proportional* measures. We now propose to lay off upon such perspective lines parts *equal* to parts taken upon a line of *real* measures. Any triangle will do to transfer proportional parts, but to

transfer equal parts we must have an isosceles triangle; for it is only in isosceles triangles that the parts into which the adjacent sides are divided by lines drawn parallel to the base are equal, each to each.

Lines of equal measures.

This is illustrated by Fig. 11, in which the line *m m*, at the top, is the line of horizontal measures. The actual dimensions of the sides of the house and of the doors and windows are laid off on this line, and connected with the inclined lines of the plan by means of lines drawn parallel to the base of an isosceles triangle.

99. It is plain that what is here done in the orthographic plan could be done in a perspective plan if we knew in what direction to draw these parallel lines; that is to say, if we could find the vanishing point of the base of the isosceles triangle.

The Point of Distance, again, as a point of equal measures.

And this is, in fact, very easy, for a simple inspection of the figure shows that the *point of distance* is the auxiliary vanishing point in question. If the spectator at S looks in the direction of the parallel lines by which the right-hand line R is divided, he will see D^R; and in like manner D^L is the vanishing point of the parallels by which distances taken on the line of horizontal measures are transferred to L.

And that this is as it should be is plain from a further inspection of the figure. For the sides of the isosceles triangles at the top are by construction parallel to the sides of the triangles $SV^R D^R$ and $SV^L D^L$. These last are accordingly isosceles too, and their two long sides should be equal. The auxiliary vanishing points

then, should be just as far from the vanishing points as these last are from the station point; as the points of distance are, by construction. $V^R D^R$ was originally taken equal to $V^R S$, and $V^L D^L$ to $V^L S$ (89).

100. The points of distance, then, are the vanishing points of the parallel lines which will intercept upon a perspective line parts equal to those intercepted upon its line of measures.

This is illustrated in Fig. 12, where, in the perspective plan, parts laid off by scale on the ground line, or line of horizontal measures, are transferred in their true dimensions to the perspective lines R and L. In this way the length of the walls and the position of the doors and windows is exactly determined. These dimensions being already shown at the given scale in the little elevations, it suffices to transfer them directly from those drawings to the ground line with a measuring strip.

The Method of the Perspective Plan.

In this way a complete and accurate *perspective plan* can easily be constructed; the length of all horizontal lines and the position of all vertical lines will then be known. The length of vertical lines, which gives the position of horizontal ones, is easily obtained, as we have seen, from vertical lines of measures.

The second perspective plan, above the other, gives the plan of the roof and dormers.

Fig. 11.

The Method of Direct Projection.

101. Fig. 11 affords an alternative method of obtaining the horizontal dimensions; that is to say, the position of the vertical perspective lines. If we again regard the plan at the top as the

plan of a miniature house, or model, set six inches from the eye at S, and regard *m m* as the plane of the picture, in contact with it, we can, by drawing lines from every point in the plan to the station point, find just where every point will appear in the picture; the horizontal dimensions thus obtained can then be transferred directly to the picture in Fig. 12. They are shown by marks on the lower side of *m m*, and will be found to agree exactly with the dimensions obtained from the perspective plan.

This method, which is called that of *direct projection*, is often more convenient than the other, especially when the orthographic plan has previously been prepared, and when, as in the present case, the subject is simple.

Advantages of the Perspective Plan.

But the method of the *perspective plan* is more convenient for designing in perspective, or for making a perspective drawing, as often has to be done, from mere sketches. It takes up less room, in the vertical direction; it is less laborious, though requiring perhaps more knowledge and skill; and it has the advantage previously pointed out, — that it enables the position of points at different levels to be separately determined, by the use of separate perspective plans, and enables several successive drawings to be made, if necessary, without repeating the bulk of the labor, since the perspective plan can be made on a separate piece of paper, and used more than once.

Moreover, the points established on a perspective plan explain themselves: it is clear at a glance, and after any lapse of time, which denotes the door, which the

window. In working by direct projection from the orthographic plan, on the contrary, it is almost impossible to remember which point is which, and much time and labor are lost by this confusion.

The *modus operandi.*

102. In making a perspective drawing, we generally fix first the points V^R and V^L as far apart as is convenient, then strike a semicircle between them, and take the point S in such a position upon it that S V^R and S V^L shall be parallel to the sides of the object to be drawn. The points of distance and other vanishing points, and the various horizons, can then be determined; the perspective plan drawn wherever is most convenient, and the perspective picture erected above it by the aid of vertical lines of measures. It is to be noticed that, since the vanishing points and traces depend solely upon the *direction* of the lines and planes, not upon their *position* (11), the building may be set to the right or to the left of the centre, below or above the Horizon, according as more or less of the left-hand side or of the roof is to be shown.

Care must be taken not to confound the point S with the point assumed for the front corner of the perspective plan, which is entirely independent of it. The revolved lines, also, $D^R V^M$, $D^L V^N$, etc., (89) must not be confounded with the traces $V^L V^M$, $V^R V^N$, etc.; that is to say, with H L M, H R N, etc.

CHAPTER VI.

THE POSITION OF THE PICTURE. — THE OBJECT AT 45°. — THE MEASUREMENT OF OBLIQUELY INCLINED LINES.

IN the last chapter we saw how, when the real length and direction of the lines by which an object is defined are known, the real length and direction of their perspectives may be exactly ascertained. But our investigations covered only the cases of lines parallel to the plane of the picture, whether horizontal, vertical, or inclined, and of horizontal lines inclined to the picture. It remains to find out how to apply a scale to lines inclined to the picture, that are not horizontal, such as the lines of the gables, and of the hips and valleys of the roofs.

Plate V. shows how this is done, and at the same time illustrates some other points of interest. In this plate, as in the previous plate, we have at the side a plan, on a small scale, illustrating the relative position of the spectator at the station point S; of the object, at A; of the plane of measures, at *m m*; and of the plane of the picture, at *p p*. In the present plate the position of the object at A, and of the spectator at S, is the same as in Plate IV.; but the position of the plane of the picture and of the plane of measures is changed, their previous position being indicated by dotted lines. Their

Plate V.

position is now taken so that the centre at V^C, and the vanishing point of 45° at V^X, coincide. That is to say, the principal lines of the building, R and L, at right angles with each other, are now at 45° with the plane of the picture, and consequently with the plane of measures; and the line X that divides the angle, making 45° with each, is accordingly at right angles with the picture and the plane of measures, and is parallel to the Axis. V^X, then, of course, comes in the same place that V^C does.

103. Now, in the first place, this illustrates the point that, the position of the spectator and of the object being given, it is purely a matter of convenience how we take the plane of the picture, and in what direction its axis is drawn. In Plate IV., Fig. 11 and Fig. 12, the axis was directed towards the end of the house, V^C coming near the corner, and the left-hand side of the house was less inclined to the plane of the picture than the right-hand side. In Plate V., Fig. 14 and Fig. 15, the axis is directed further to the left, V^C coinciding in position with V^X, and the plane of the picture and plane of measures are equally inclined to both sides of the house, making 45° with each. But of course this would not change the appearance of the object A, as seen from S; and if the drawings were both made so as exactly to cover and coincide with it, the two representations would look exactly alike when viewed each from its own station-point, so far as concerns their main outlines. These are not affected by the alterations and

Position of the plane of the picture.

Fig 14.

Fig. 15.

repairs the house has undergone since we last saw it.

104. In the second place, the plate illustrates the point that it is a much simpler thing to make a drawing in perspective when the plane of the picture, as in Fig. 15, is at an angle of 45° with both sides of the object than when it lies accidentally, as in Fig. 11.

The object and the picture at 45°.

For this attitude of the object sets the station-point at its maximum distance from the picture, which is obviously an advantage. This distance, S V^C, is in this case just half the length of the Horizon between V^R and V^L. Moreover, since S is equidistant from these points, and V^C, which is also V^X, is half way between them, it follows that D^X coincides with V^R, and also that there is another D^X coinciding with D^L; that D^L and D^R are also equidistant from V^C; and, in short, that the whole series of vanishing points and points of distance is symmetrical about H P P′ in one direction, as it is about the Horizon in the other.

The symmetry of the vanishing points and horizons.

105. From this symmetry, which is clearly exhibited on a small scale in Fig. 14, it follows that V^M and V^N are equally distant from the Horizon, that H L M is parallel to H R N′ and H R N′ to H L M′, and hence that V^Q and $V^{Q'}$, which lie at the intersection of these traces, are at an infinite distance. Hence the lines Q and Q′ are both parallel to the picture, the perspective of Q is parallel to H R N and H L M′, that of Q′ to H L M and H R N′, and the projections of these lines in the perspective plan are parallel to the Horizon.

106. The great convenience of this is shown in Fig. 15, where the hips of the roof of the school-house on the left, that form its outline against the sky, are drawn parallel to the horizons of the planes in which they lie instead of being directed, as in Fig. 13, to their inaccessible point of intersection.

The practical conveniences of this.

Accordingly objects are generally drawn at 45°, being set to the right or to the left of the centre, as it is desired to expose their left side or their right side more fully to view (102).

107. It will be observed that the relation of the principal distance points, D^R and D^L, to their corresponding vanishing points, V^R and V^L, is the same as that of the corners of an octagon to the corners of the square from which it is cut, the distance of each from the remoter corner being half the diagonal of the square. Compare Fig. 16 with the small plan in Fig. 14. The distance apart of the vanishing points then being as 200, the distance of the station-point from the picture at V^C will be as 100, and each point of distance will be at a distance of 141 from its own vanishing point, and 59 from the other. Or, in other words, the distance apart of the vanishing points being as 10, that of the points of distance from the centre, V^C, will be as 2, their distance from the nearest vanishing points will be as 3, their distance from each other will be as 4, and the distance of the spectator from the picture as 5, very nearly.

Fig. 16.

108. Fig. 17 shows the amount of the error involved in this assumption. It is so slight that for

Fig. 17.

practical purposes the distance from the centre to the points of distance may generally be taken as two fifths of that to the vanishing points, when the picture is at angles of 45° with the sides of the object.

109. Fig. 15 also illustrates another point of capital importance, showing how dimensions are laid off by scale upon lines lying on the *hither* side of the plane of measures; these lines are necessarily drawn to a larger scale than that used for lines in that plane, just as lines beyond it are drawn to a smaller scale. The principle of the isosceles triangle, explained in the last chapter, and again illustrated in Fig. 14, still holds good. The fence, for instance, that is built out from the front corner of the house, toward the right, has the position of its posts laid off upon the line of horizontal measures, or ground line, $g\,l$, to the right, by scale, and then transferred to the line of the fence in the perspective plane, by parallel lines drawn to D^L, the fence being in a line with the left-hand side of the house. These lines diverge as they come forward, and the scale of the fence is obviously increased. The building on the left is entirely in front of the plane of measures.

Objects nearer than the plane of measures.

110. To avoid confusion the points on the ground line intended for use beyond the plane of measures are indicated above it, and those which give the dimensions of objects this side of it, as is the case with this fence, are indicated below it. A better way in some cases is to have separate ground lines for figures in front of the plane of measures.

Obliquely inclined lines by scale.

111. Let us now take up the question left unanswered in the previous chapter, and see how dimensions can be laid off by scale upon obliquely inclined lines; upon lines, that is to say, which, like the gables and the hips and valleys of roofs, lie in inclined planes, and are hence inclined not only to the plane of the picture but also to the horizontal plane.

Line of measures lying in an obliquely inclined plane.

In the first place, it is clear that just as every horizontal plane has a line of horizontal measures, or ground line, where it intersects the plane of measures, and just as every vertical plane has a line of vertical measures which is its line of intersection with the plane of measures, so every inclined plane, such, for example, as a roof, in like manner has its own line of measures lying in it and also in the plane of measures, upon which dimensions can be laid off by scale. It is the line in which that plane intersects the plane of measures, and it passes through the point at which any line in that plane pierces the plane of measures. This line of measures and its perspective are always parallel to the horizon of the given inclined plane. For, as we have seen (78) this horizon is the intersection of the plane of the picture with a plane parallel to the given plane, and passing through the eye, while the line of measures is the intersection of the given plane, with a plane parallel to the picture. The intersections of these two sets of parallel planes must be parallel.

112. To find a line of real measures in any plane, then, it suffices to find the point at which any line in it

pierces the plane of measures, and to draw through this point, when found, a line parallel to the horizon of the plane.

113. In Fig. 15, for example, we have several such lines: *g l* parallel to the Horizon, *l m* parallel to H L M, *r n* parallel to H R N, etc. In the case of the ground line, the horizontal lines to be measured off are prolonged until they reach the plane of measures, and the line *g l* is drawn parallel to the horizon through the point thus obtained. In the case of the inclined lines of the gables, that on the end of the building at the right is brought forward till it touches the plane of measures, and the corresponding line on the porch roof of the porch of the little school-house on the left is carried back until it reaches the plane of measures. These points are ascertained from the perspective plan, the point where the horizontal projection of these inclined lines pierces the plane of measures being the projection of the point where the lines themselves do so. The lines of measures *l m*, are then drawn through the points thus ascertained parallel to the horizon of the plane L M. As these gable lines lie also in the vertical planes R Z, vertical lines of measures, *r z*, parallel to H R Z, may also be drawn through the same points.

Fig. 15.

114. In the case of the hip line, Q, of the little school-house, which is parallel to the plane of measures and to the horizon H R N (105), it is necessary to proceed as in the case of vertical lines, or any others that are parallel to the picture, and carry across it a line lying also in the plane R N, but inclined to the picture, by means of which the line where the plane intersects the

plane of measures can be found. In the figure the line of the other hip, P, is used for this purpose, its horizontal projection being found in the perspective plan. It would have done just as well to prolong the line of the eaves until it met the plane of measures, and to draw the line *r n* through the point thus found.

115. All these lines of measures lie in the plane of measures, and are consequently drawn to the same scale. On all of them one sixteenth of an inch is the true perspective representation of one foot, and any dimensions laid off on this scale can be transferred to any other lines in the inclined plane, such as the lines of the gables, by means of proper points of distance lying in the horizon of the inclined plane, just as dimensions can be transferred from the ground line to lines lying in the horizontal plane by means of points of distance taken on the Horizon.

Manifold points of distance.

116. The different points of distance upon these inclined horizons, are easily obtained from those upon the horizontal Horizon, if we may use such an expression. It is obvious, indeed, that the problem is a simple one, if we remember that the point of distance corresponding to any vanishing point is found by setting off upon the horizon passing through that point the true distance of that point from the station-point. If this is done, the triangle S V D formed by the three points will, of course, always be isosceles by construction. Now, the distance of every vanishing point from the eye is known, and can readily be laid off on all the traces, or horizons, that pass through it.

117. The distance of V^R from S, for example, is known to equal V^R D^R, and this distance laid off on H R N, H R N′, H R Z, and the traces of any other planes that contain R, give the corresponding points of distance on those horizons, D^{R1}, D^{R2}, D^{R3}, D^{R4}, etc., all, in fact, lying at equal distances about V^R on the circumference of a circle whose radius equals V^R S, the distance of the vanishing point in question from the eye. This is shown both in Fig. 14 and in Fig. 15. In like manner a circle may be drawn about V^L, containing the distance-points that belong to that vanishing point. Its radius will be V^L D^L, the distance of V^L from the eye.

The *locus* of points of distance.

Fig. 14.

Fig. 15.

Such a circle is the *locus* of the points of distance.

118. The distance of V^M from the eye is V^M D^R, as is obvious if we suppose the triangle V^M D^R V^R revolved back around its side H R Z until D^R is at S again. A circle drawn around V^M, with V^M D^R=V^M S as a radius, will accordingly give points of distance, by which any inclined line M, vanishing at V^M, may be measured off by lines of measures lying in any of the planes that contain it, and whose horizons accordingly pass through V^M.

As we have taken the lines M and M′ at an angle of 30° with the Horizon, the angle M D^R M′ is 60°, the triangle M D^R M′ is equilateral, and D^M falls upon $V^{M\prime}$, and $D^{M\prime}$ upon V^M.

In every case, of course, the line of real measures, like the lines of proportional measures discussed in a previous chapter, is drawn parallel to the horizon of the plane containing it (76).

119. It will be noticed that there are two points of distance on H R Z, at equal distances above and below V^R, just as we found that both V^R and V^L would serve as points of distance to V^X. Either of these can be used instead of D^R to set off given lengths on the right-hand horizontal lines. In the upper perspective plan, for example, the length of the right-hand side of the house is set off not only on the ground-line, and transferred by means of D^R, but also on the vertical line of measures, both above and below the ground, and transferred by means of D^{R2} and D^{R3}, giving in every case the same points.

Two points of distance on each horizon.

Indeed, it is practicable to have a second point of distance on all the horizons beyond V^R. They would be the vanishing points of the bases of *obtuse-angled* isosceles triangles. Such a triangle is shown in Fig. 18, *b*. Its employment is shown in the upper perspective plan, where the length of the right-hand side of the house is laid off by scale on *g l*, to the *left* of the corner, and transferred to the perspective line by means of the point D^{R4}, which is out of the picture.

Fig. 18.

120. If, then, through the point at which any perspective line pierces the plane of measures, a line of measures be drawn parallel to the horizon of any plane that contains the line (that is to say, parallel to any horizon passing through the vanishing point of the line), any distance laid off by scale on the line of measures may be transferred to the perspective line by means of a point of distance in that horizon; and this point of distance will be the point where that horizon is cut by the

circle, which is the *locus* of the points of distance, — a circle whose radius equals the distance of the vanishing point of the perspective line from the eye, and whose centre is at the vanishing point.

This is illustrated in the lower perspective plan, in which a certain distance, being that from the corner of the house to the further edge of the window, is laid off on several lines of measures, *g l*, *l z*, and *l m′*, parallel to the several traces that meet at V^L. Lines drawn to the points D^L on those traces all give the same point upon the line L.

121. As the points taken upon the line of measures are all equidistant from the corner of the plan, they lie in the circumference of a small circle, which is the *locus* of the points of measures, just as the corresponding points of distance lie in the circumference of a large circle; and we have the curious phenomenon of lines drawn from certain points on the small circle to corresponding points on the large one, all intersecting at the same point, upon a line joining the two centres. But Fig. 19, which presents these geometrical relations in a diagram, shows that there is nothing surprising in this.

The *locus* of points of equal measures.

Fig. 19.

122. Now, just as any line drawn in any direction, at random, through one end of a perspective line, may be used as a line of proportional measures (81), so any such line that lies in the plane of measures may be taken as a line of real measures; and any dimensions taken upon it by scale may be transferred to any perspective line that touches it,

Random lines of equal measures.

by means of a horizon drawn through the vanishing point of the perspective line, parallel to the random line, the point of distance being taken upon that horizon at the same distance from that vanishing point as its other points of distance.

This also is illustrated in the lower perspective plan, where, in addition to the lines of measures drawn parallel to the horizons already obtained, a line of measures, *l t*, is drawn at random at an angle of 15°. A new horizon, H L T, is drawn through V^L parallel to it, and a new point of distance, D^{L5}, obtained upon it, where it intersects the *locus* of D^L.

The same thing is done again in the picture Fig. 15, where the divisions of the fence, on the right of the house, are taken by scale on an arbitrary line of measures just above. As this also is taken at 15°, the same point of distance, D^{L5}, serves to transfer the dimensions to the line L, the top line of the fence.

Fig. 15.

Perspective lines as auxiliary horizons.

123. Finally, just as of any two perspective lines having the same vanishing point, one may be regarded as the horizon of a plane passing through the other (82), so a point of distance taken upon the first, at the proper distance from that vanishing point, may be used to lay off given dimensions upon the second, by using a scale upon a line of real measures drawn parallel to the first, from the point where the second pierces the plane of measures.

This is illustrated in the plan of the little schoolhouse shown in the plate. The front line of the plan is

regarded as the horizon of a plane, and upon this horizon the point of distance D^{R5}, upon the *locus* of D^R, is easily determined. A line of measures is drawn parallel to it through the further corner of the plan, where the line R on the back of the building touches the plane of measures. The true width of the building being laid off on this line of measures by scale, and transferred to the perspective of the further side by means of a line drawn to the point of distance, we obtain the left-hand corner of the building, and the position of the rear window, as we ought to.

124. Here, as before (84), we must be careful not to suppose that this line of measures is drawn on the floor of the school-house. It lies, in fact, in the vertical plane of measures, being an inclined line parallel to the plane of the picture. So, also, the point of distance is not on the ground, but is in the infinitely distant horizon which the front line of the building covers and apparently coincides with.

Proportional measures, equal measures, and scale measures.

125. It is to be observed that the methods of measuring off perspective lines described in this and the preceding chapters are strictly analogous to the method of dividing perspective lines described in the previous chapter. By that method, called the Method of Triangles, the perspective of a finite line is divided up, in any desired proportion, by drawing from one end of it a line, called a line of proportional measures, parallel to the plane of the picture, and then setting off upon this line, at any

convenient scale, dimensions proportional to those desired. These are then transferred to the perspective of the given line by means of a point of proportional measures which is taken upon the horizon of the plane in which both lines lie. This horizon passes through the vanishing point of the given line, and the point of proportional measures is the vanishing point of the base of the scalene triangle, the other two sides of which are the given line and the line of measures.

By using the point of distance, however, as a point of measures, the triangle becomes isosceles, and the parts cut off upon the perspective line are not only proportional but equal to those taken upon the line of measures. It is now the line of measures that is of fixed dimensions, and the perspective line is not divided up, but has these dimensions set off upon it, being treated as a line of indefinite length. In the Method of Triangles the line to be divided is of definite length, the line of proportional measures is an indefinite line, and the point of (proportional) measures may fall anywhere in the horizon of the plane of the triangle. In measuring a given distance upon a perspective line, on the other hand, the line of equal measures, or of scale measures if a scale is used, is a definite portion of the ground line, or other line of measures, the given line is of indefinite length, and the point of (equal) measures is the point of distance.

In dividing a given line proportionally, the length of the line is indeed always fixed, and that of the auxiliary is indefinite; in transferring real dimensions, on the other

hand, these fix the length of the auxiliary, and it is upon an indefinite length of given line that they are to be set off. This difference is illustrated in Fig. 18, Plate V. In *a* the given line is of definite length, the auxiliary of indefinite; in *b* the contrary is the case.

Moreover, it is to be observed that although a line of equal measures, like a line of proportional measures, may be drawn parallel to the picture from any point in the perspective line, it will not serve for scale measurement unless it lies in a plane of measures whose position has been already determined. It is accordingly necessary to extend the given line until it touches or pierces the established plane of measures, before a line of scale measures can be drawn, as has frequently been exemplified in this chapter. The difference is illustrated by comparing the treatment of the gable in Plate III. with that of the gable in Plate V. In the latter the line of the gable is extended so as to bring the lines of measures into the same plane as all the other lines of measures.

CHAPTER VII.

PARALLEL PERSPECTIVE. — CHANGE OF SCALE.

THE last chapter discussed the case in which the plane of the picture, and consequently the plane of measures parallel to it, is set at *an angle of 45°* with the sides of the object, so that the "vanishing point of 45°," V^X, coincides with V^C, the centre of the picture, and the principal vanishing points of the right-hand and left-hand lines, V^R and V^L, and their points of distance, D^R and D^L, are symmetrically disposed on each side of it. The diagonal Y, at right angles to X, is in this case, of course, parallel to the picture, and to the Horizon.

126. Let us now consider the analogous case in which the plane of the picture, and consequently the plane of measures, is taken parallel to one of the principal sets of lines, and at right angles to the other. This is illustrated in Plate VI. When objects are thus represented with one side parallel to the picture, and the adjacent sides perpendicular to it, they are said to be drawn in Parallel Perspective.

Plate VI.

127. The relation between this case and that discussed in the last paper is shown in the two buildings upon the quay, on the left-hand side

Fig. 20.

of Fig. 20. The nearest one, whose roof rises just above the rail of the descending steps, stands at 45° with the picture, just as in Plate V., having the vanishing points of its main lines at V^R and V^L; while one set of its diagonals, X, as seen in the perspective plan below, converges at V^C, and the other, Y, is parallel to the Horizon; the lines of the hips, P and P′, being directed to V^P and $V^{P'}$, and the hips Q and Q′ being parallel to the horizons H R N and H L M (105). The points V^M and V^N are off the picture, but V^P suffices to determine the position of these horizons. These horizons are not shown in the picture, to avoid confusion. The diagonal X here coincides with C.

128. The larger building beyond, like all the other objects in the picture, is drawn in Parallel Perspective. It is set at an angle of 45° with the building just mentioned, the sides of one being parallel to the diagonal lines that divide the angles of the other, and *vice versa.* This is rendered more obvious by comparing their perspective plans.

Parallel Perspective.

129. These plans, it will be noticed, are drawn wherever it is most convenient to put them, that of the further building being taken so very far below the ground as to come lower down on the paper than that of the nearer building. The diagonal X here coincides with R.

130. Now, it is to be observed that while the planes of the nearer roofs, as in the previous plates, have for their horizons the lines H R N, H L M, etc., and the hips have their vanishing points at V^P, $V^{P'}$, etc., the roof of the further building, and the tops of the posts in the foreground, which,

Inclined planes, with one element parallel or normal to the picture.

for convenience, are given the same slope, present a new case, which we have not hitherto met, — the case of inclined planes whose horizontal element is either parallel or perpendicular to the picture.

131. The several flights of steps, also, ascend and descend along inclined planes, either at right angles to the picture, or, as is the case of those at either end of the platform in the extreme foreground, parallel with it. Let us take these flights of steps first, and make the rise of each step six inches, and the tread nineteen inches. Now, as the flights in the foreground are parallel to the picture, they will be drawn in their true proportions, and will give the true slope of the steps; and if we suppose the plane of measures to coincide with the front of the little pavilion, or with the further end of the steps, we can lay off the steps by scale at once, and ascertain their true slope. It proves to be about 20°.

132. To ascertain the point V^A, which determines the direction of the sloping lines of the flights which ascend at right angles to the picture, we have only to draw a line at $V^L, = D^C$, making an angle of 20° with the Horizon. Its point of intersection with H P P′, above V^C, will be the point in question (88). $V^{A\prime}$, the vanishing point of the descending flights, will be at an equal distance below V^C.

133. The horizons of the inclined planes in which these steps lie, pass, of course, through the vanishing points of all the lines that lie in them (13, II.). V^A and $V^{A\prime}$ are the vanishing points of the steepest lines of the ascending and descending planes, their horizontal

element K being parallel to the picture, and their vanishing points being at an infinite distance to the right and left. The horizons drawn through V^A and $V^{A'}$ are accordingly drawn horizontal, or parallel to the Horizon. The horizon of a plane is, indeed, always parallel to that element of the plane which is parallel to the plane of the picture (76); for that element has its vanishing point on the horizon, but at an infinite distance. These two horizons are H K A and H K A′, just as the Horizon is lettered H R L.

134. The horizon of the inclined planes of the flights of steps that ascend to the right and go down to the left, on the edges of the picture, passes of course through the point V^C, the vanishing point of their horizontal element, and is parallel to the steepest line of the slope, S, which is the element parallel to the picture.

135. The horizons of the inclined planes of the roof of the building on the left, and of the little flat pyramids on top of the posts, are determined in the same way. The slope of these planes is about 25°; they are accordingly steeper than the slope of the steps; and the vanishing points $V^{A}{}_{2}$ and $V^{A}{}_{2}{}'$, found by drawing lines from V^L or V^R at an angle of 25°, are further from the Horizon than V^A or $V^{A'}$. H K A_2 and H K A_2' are the horizons of the front and back planes, while the planes of the right-hand and left-hand sides pass diagonally across the picture through V^C at an angle of 25°. They are lettered H C D and H C S^2. The vanishing points of the hips, or angles of the pyramids, are at the intersections of these

horizons with $H K A_2$ and $H K A_2'$ at the points marked V^{M2}, V^{N2}, $V^{M2'}$, $V^{N2'}$. They are, of course, in the horizons $H L Z$ and $H R Z$, since these hips and angles obviously lie in vertical planes making 45° with the picture.

136. The only other object in the figure, the pavilion or belvedere, a little building just twice as long within as it is wide, is easily drawn. All the lines of the perspective plan are either parallel to the Horizon or converge to V^C, and are cut off at the length required by setting off the true length by scale, upon $g\,l$, and transferring it to the perspective line by a line drawn to the point of distance, $V^R, = D^C$. The intermediate points are determined in the same way. The front of the building being in the plane of measures, all its parts are drawn to scale, proportionally to their real dimensions, and the same is true of the further end of the building and of the arched wall in the middle, only that being more distant, the scale on which they are drawn is smaller. All the arches are struck from the perspective of their centres, which, being in reality all on a horizontal line perpendicular to the picture, occur in the perspective on a line drawn from the centre of the front arch to V^C. Their exact position on that line is determined on the perspective plan.

Dimensions.

In the same way the rafters in the roof may be laid off exactly, two feet apart, using the point of distance $D^C = V^R$, or, if their number is known, the space they occupy may be divided into six equal parts by using the method of triangles, described in the fourth chapter.

137. For the length indicated in the figure, and with the station point so near the picture, there is no practical inconvenience in thus using V^R or V^L as points of distance. But if, as might easily be, the room to be drawn were twice as long as this, the point *g* would be inconveniently far away; and if the station point were farther from the picture, — and it is always an object to have the station point as far away as the point from which the picture will generally be regarded, — say two feet, the point V^R also would be practically inaccessible.

Change of Scale.

138. These inconveniences, which are likely to occur in oblique perspective as well as in parallel perspective, may in all cases be got over by substituting another triangle for the isosceles triangle hitherto employed. Instead of using a triangle whose legs are equal we may just as well employ a scalene triangle, provided only the ratio of the two legs is known. Lines drawn parallel to the base will not now indeed divide the adjacent sides into equal parts, but we can just as easily as before cut off any required dimensions.

139. This is illustrated in Fig 21, *c*. We have here, as in Fig. 12 and Fig. 14, the plane of the picture *p p* in immediate contact with the object, which is here the model of a small room or passage, whose plan, as above, occupies two squares. Let us suppose the spectator to be at S_1, at a distance from the picture of S_1 D^C, the length of the Axis, C^O. V^R, then, or D^C will be at an equal distance to the right, as shown, on the prolongation of *p p*. This gives the

Fig. 21.
Fig. 12.
Fig. 14.

right-angled isosceles triangle, $S_1 V^C D^C$, and the length of the room laid off on *p p* in the other direction may be transferred to the side of the room by the line D *d*, drawn parallel to $S_1 D^C$. Fig. 21, *b*, shows this done in perspective, determining the point *d*, as just now in Fig. 20.

Fig. 20.

140. But this point *d* will be fixed with equal precision if we take instead of D^C another point, $D_{\frac{1}{2}}$, half way from V^C to D^C and make the base of our triangle $S_1 D_{\frac{1}{2}}$. The triangle is no longer isosceles, but we know that lines drawn parallel to the base will make the segments of the short side just half as long as those of the long side, and that does just as well. For if we now lay off in the other direction, just as before, half the length desired, a line drawn parallel to $S_1 D_{\frac{1}{2}}$ will give *d*, just as before. And in like manner we might take $D_{\frac{1}{4}}$, half way between V^C and $D_{\frac{1}{2}}$, and lay off one quarter the required length of the room, etc., with the same result.

141. Applying this now to Fig. 20, by taking $D_{\frac{1}{2}}$ or $D_{\frac{1}{4}}$, measured off along the Horizon from V^C at half or quarter the distance of the station point from that vanishing point, we can cut off any desired dimensions on the perspective lines that converge at V^C by laying them off upon the ground line, *g l*, by a scale of equal parts half or quarter as large as those used in the plane of measures, and employed for the horizontal and vertical lines in that plane. Instead of using a half-inch scale, and using D^C as the point of distance, we may use a quarter-inch scale, and transfer the dimen-

Fig. 20

sions to C by means of $D_{\frac{1}{2}}$, or an eighth scale and use $D_{\frac{1}{4}}$.

142. This results may be summed up as follows:—

Points of half distance, etc.

Auxiliary points of distance, which may be called points of half distance, quarter distance, etc., may be obtained by laying off from the vanishing point of any line, upon any horizon that passes through that vanishing point, a half, or a quarter, etc., of the real distance of the station point from that vanishing point; the required dimension must then be laid off in an opposite direction upon a line of half or quarter measures, etc., drawn parallel to that horizon through the point where the perspective line in question touches the plane of measures, the scale employed being a scale of equal parts, half, or quarter, etc., as large as those employed for lines in the plane of measures.

143. It is sometimes convenient also to use a smaller scale for lines parallel to the plane of measures and at some distance behind it, whether vertical, horizontal, or inclined. This is tantamount to establishing another plane of measures, and another ground line, two, three, or four times as far off, and diminishing the scale used in the picture accordingly.

144. This use of points of half distance, third distance, or quarter distance, and this employment of an auxiliary plane of measures, and the change of scale involved in both these devices, are obviously just as practicable in other cases as they are in this. But they are most often used in the case of parallel perspective. For in oblique perspective the need of having the vanishing

points within convenient distance generally limits the distance of the station point and keeps the points of distance near at hand. Both are always nearer than the remoter vanishing point. But in parallel perspective one of the principal vanishing points is at an infinite distance, and the points of distance, though nearer than that, may yet be quite out of reach. The use of points of half and quarter distance, etc., enables one to set the station point as far away as he pleases. There is absolutely nothing to prevent his taking the point of view most favorable for his purpose, the point of view, namely, that will give the best proportions to his picture.

An inverse procedure common.

145. In practice it is most convenient *to assume the desired proportions* at the outset; that is to say, having determined on the scale at which the nearest end of the street or room is to be drawn, to make the further end as large, and to set it as far on one side and as far up or down, as will look best. Vanishing lines drawn through the corresponding points of the two ends will then determine the centre, V^c, and the Horizon. This is all that need be determined, since the length of the room or street is supposed to be known. If from the near end of the perspective of this length, the real length is laid off upon the ground line, at any convenient scale, and the last point connected with the further end of the perspective line, and prolonged until it meets the Horizon, the point thus ascertained will be a point of half, quarter, or third distance, according as

the scale chosen is a half, a quarter, or a third of that used for the plane of measures. The corresponding distance of the station point in front of the centre, V^C, will then be two, three, or four times the distance of the centre from this auxiliary distance point.

The two views of the street in Fig. 22, both of which were drawn in this way, illustrate the importance of carefully proportioning the parts of the picture. The upper one shows the street as it would appear quite near at hand, much reduced; the other, drawn to the same scale, shows how it would look at a greater distance. Fig. 22.

146. Inasmuch as the relative size to be given to the two ends of a room, drawn in parallel perspective, or to the two ends of a street, depends thus entirely on the position of the spectator, and not at all on the real length of the side, it follows that a long room seen from one point may be drawn to look just like a short room seen from a nearer point, and that there is no knowing which is which. This is illustrated in Fig. 21, where the front half of the room in the upper figure, *a*, as seen from S_2 is just the shape of the whole room in the lower figure, *b*, as seen from S_1. The same drawing also, if regarded from different distances, may give the impression of a long street or room, or of a short one. Fig. 21.

147. In sketching interiors on the spot, the point of view is generally excessively near. Such sketches, when viewed from a greater distance, generally give too great an impression of length, and often have to be re-

drawn, so as to show the room as it would look from a point which really is outside of it.

Practical Limitations in the use of Parallel Perspective.

148. Parallel perspective is not often used for a single object, inasmuch as in order to show a second side, at right angles to the picture, it is necessary to set it a good way from the centre. Fig. 23 shows the top of a chimney, the end of which is parallel to the picture, while the long side is perpendicular to it. The horizontal lines of the brickwork on this side are directed to V^C, those on the end are parallel to the Horizon, while the vanishing lines of 45°, as seen on the perspective plan below, are directed to V^L and V^R.

Fig. 23.

149. It is to be noticed that all the surfaces on the end of the chimney are drawn of their true shape and proportion, as if seen in elevation. Still the whole end of the chimney is not drawn in elevation, the relations of the several parts being changed and the symmetry of the whole disturbed, since the nearer surfaces are set further to the right and higher up, and since something is seen of the horizontal surfaces that separate them, which in the elevation just below, drawn in orthographic projection, are not seen at all. The chimney certainly looks very ill drawn, and it is not easy, even by keeping the eye sedulously at the station point, opposite V^C, to make it look quite right.

150. The use of parallel perspective is accordingly pretty much confined to cases where two objects are to be shown, one on the right and one on the left, as in

street views, or interiors. In these cases the eye naturally takes a central position, opposite the middle of the street or the middle of the room represented. It is not necessary, of course, that the Axis should be exactly in the middle, and it is generally taken near one side, so as to show as much as possible of the other, and thus prevent an absolute symmetry.

Notation. Of the lines which are parallel to the picture, the vertical ones, going to the zenith, are lettered Z; the horizontal ones are lettered K, instead of H, which would be confusing; those which incline down to the right are lettered D, for *dexter*, as in heraldry, and those which incline down to the left S, for *sinister*. Those which slope directly backward, up or down, are lettered A and A′, for *altitude*, as in astronomy. The planes normal to the picture are lettered, accordingly, C Z, C K (= R L), C D, and C S, and the horizons H C Z, H C K (= H R L), H C D, and H C S, each plane being designated by its horizontal element and its line of greatest inclination, as usual.

CHAPTER VIII.

OBLIQUE, OR THREE-POINT PERSPECTIVE.

“One-Point” or Parallel Perspective.

151. THE last chapter discussed the phenomena of Parallel Perspective, in which, of the three sets of lines that define a rectangular object, two are parallel to the picture and have their vanishing points accordingly at an infinite distance; the third alone has its vanishing point in the plane of the picture. This may be called, accordingly, “One-Point Perspective,” since it employs only one vanishing point, V^X at V^C.

“Two-Point” or Angular Perspective.

152. In the previous chapters only one of the principal sets of lines, namely, the vertical lines, were parallel to the picture, both sets of horizontal lines being inclined to it at an angle, one to the right and one to the left. This may accordingly be called “Angular” or “Two-Point Perspective,” two vanishing points being employed, V^R and V^L.

“Three-Point” or Oblique Perspective.

153. We now come to a third case, that in which all three of the principal lines of a rectangular object are inclined to the picture, the object presenting towards the eye a solid corner. In this case, all three vanishing points are employed, and the drawing may be said to be made in “Ob-

lique," or "Three-Point Perspective." Plate VII. illustrates this case, Figs. 24, 25, and 26 presenting examples in which, though the object is vertical, the plane of the picture is inclined; while in Fig. 27 the picture is vertical, as usual, but the cubical block on the floor, the two covers of the box in the foreground, and the chair, are all tipped so that all their edges are inclined to the picture. They are accordingly drawn in Three-Point Perspective. Plate VII. Figs. 24, 25, 26. Fig. 27.

154. Fig. 24 shows a post at the corner of a fence as it appears when one looks down upon it, the plane of the picture being inclined backwards at the top. Fig. 25 is a drawing of the tower of old Trinity Church, in Boston, which was destroyed by the fire in November, 1872, as it appeared when one was looking up at it, the top of the picture being inclined forward. Fig. 26 is a similar view of the tower and spire of Salisbury Cathedral, taken from a photograph. The vanishing points in Fig. 24 are at V_1, V_2, and V_3; those of Fig. 25 at V_4, V_5, and V_6; and those of Fig. 26 are not shown, but can easily be found. Fig. 24. Fig. 25. Fig. 26.

155. To make these drawings look *natural*, the paper should be held at an angle, below the eye for the first and above the eye for the other two. The vanishing points of the vertical lines, V_3 and V_6, should be just above or below the eye.

156. Fig. 27 illustrates all three kinds of perspective, the room being drawn in parallel perspective, with only one vanishing point, at V^C; the bookcase and the box in angular perspective, with two vanishing Fig. 27.

points, at V^R and V^L; and the lids of the box, with the chair and the cubical block, in Three-Point Perspective, with vanishing points at V^L, V^M, and V^O. The three sets of planes, as marked on the cube, are of course L M, L O, and M O; and their traces, H L M, H L O, and H M O, form a triangle lying between the three vanishing points.

157. A plane of measures is supposed to pass through the nearest corner of the cube. In this lie three lines of measures, *l m*, *l o*, and *m o*, parallel to the three traces (79). On each of these lines the real length of the edge of the cube is measured off, giving the six points *l*, *l*, *m*, *m*, *o*, *o*. These dimensions are transferred to the three edges of the cube, L, M, and O, by means of the points of distance D^L, D^M, and D^O, which indicate the distance of each of these vanishing points from the station point, S, in front of the picture, opposite C. Each of these points of distance occurs twice, once on each of the horizons that meet at its vanishing point (120).

158. In the same way the width of the box is laid off on a vertical line passing through its front corner, and transferred to its right-hand edge by means of D^R, in the trace H M O or H R Z. Half the width of the box, which is the width of each half of the cover, is transferred to the inclined lines of the cover, directed to V^M and to V^O, by means of the points of distance, D^M and D^O, on H M O.

159. This is all exactly in accordance with what has

been done in previous cases, and involves no new principle. The only new question which oblique perspective presents relates to the position of the station point, S, of the centre, V^C, of the various points of distance, and of the three vanishing points. Their relations are obviously much more strictly defined than in the previous cases. For, in One-Point Perspective, the vanishing point, V^C, being given, the station point may be anywhere upon the Axis, the line passing through V^C in a direction perpendicular to the plane of the picture. In Two-Point Perspective, the vanishing points V^R and V^L being given, the station point, S, must be somewhere on the circumference of a semicircle, whose diameter lies between those points, and which is itself in a horizontal plane perpendicular to the plane of the picture. But any point in this semicircle will do. In Three-Point Perspective, it must in like manner lie somewhere in the circumference of each of three semicircles, whose diameters are the three sides of the triangle formed by the three vanishing points. For, since the three edges of the rectangular object form right angles with each other, the lines drawn from the eye to the vanishing points parallel to those edges must also be at right angles with each other. These three lines, in fact, together with the three horizons, form a triangular pyramid, the vertex of which, at S, is composed of three right angles. This pyramid is of the same shape, obviously, as the small triangular pyramid that would be formed by cutting across the corner of the object rep-

The problem of the Station Point and of the Centre of the Picture, the three Vanishing Points being given.

resented with a plane parallel to the plane of the picture. The lines of intersection in each plane would of course be parallel to the horizon of that plane (78). The corner of the cube in Fig. 27 is represented as cut across in this way.

Fig. 27.

160. Now it is obvious that only one such pyramid can be constructed upon a given triangle as a base; that is to say, given the three vanishing points, the position of the point S is absolutely fixed; there is only one point at which the eye can be placed and find each pair of vanishing points 90° apart. Fig. 28 illustrates this, the semicircles that contain the three right angles being foreshortened into ellipses.

Fig. 28.

161. Another way of regarding the problem is to consider that, since the plane in which each semicircle lies is not perpendicular to the picture, but is inclined to it at an unknown angle, the position of the station point is really limited only by the condition that it must lie somewhere in the surface of a hemisphere of which the given horizon is a diameter. As this is true for each of the three horizons, the station point must be a point common to the three hemispheres. Now three hemispheres, whose diameters form a triangle, can have but a single point in common. Two of them will intersect each other in a semicircle perpendicular to the plane of the triangle, and the point where this semicircle is cut by the third hemisphere will be the point in question. Fig. 29 illustrates this view of the subject.

Fig. 29.

162. It is plain from an inspection of the figure, and of the little figure alongside, that the small semicircles in

which these hemispheres intersect will be projected as straight lines at right angles to the lines connecting their centres. But as the lines connecting the three centres are obviously parallel to the three diameters, it follows that the three straight lines in which these three semicircles are projected, and which meet in the point V^C, the projection of the apex of the pyramid, are drawn from the corners of the base perpendicular to the opposite sides. This affords an easy method of determining the point V^C.

To find the Centre.

163. This proposition, that the projection of each edge of the pyramid is perpendicular to the opposite side of the base, is, in fact, merely an illustration of the familiar proposition that if a line is normal to a plane, its projection upon a second plane intersecting the first is perpendicular to the line of intersection. Each edge of the pyramid is obviously normal to the opposite face of the pyramid, and its projection upon the base must accordingly be perpendicular to the opposite side of the base, where the face of the pyramid cuts it.

164. *Note.* This is not the place to demonstrate the proposition, of which the demonstration is to be sought in the treatises on plane geometry, that perpendiculars let fall from the vertices of a triangle upon the opposite sides will meet at a point. But it is worth while, perhaps, to observe that this point of symmetry within the triangle is only one of four such points, the others being (*a*) the centre of the inscribed circle, (*b*) the centre of the circumscribed circle, and (*c*) the centre of gravity. Fig. 30 *a*, *b*, *c*, *d*, exhibits a comparative view of these four points.

Fig. 30.

165. The point V^C being thus ascertained, it only re-

mains to determine the height of the pyramid, that is, the distance of the station point, S, in front of the picture, and the length of the three edges of the pyramid, that is to say, the distance of the three vanishing points from the station point.

To find the Station Point.

Fig. 31 shows how these distances may be determined. A plane perpendicular to the picture is passed through either edge of the pyramid. Its intersection with the opposite face and with the plane of the picture, or base of the pyramid, will form a right-angled triangle. This triangle, when revolved about its hypothenuse into the plane of the picture, will give V S, the length of the edge in question, and the height of the pyramid, or distance of the eye from the picture, V^c S. This operation is repeated on a larger scale in Fig. 27, giving S_1.

Fig. 27.

166. Fig. 28 shows how the distance of the eye from two vanishing points, that is to say, the length of the two edges of the pyramid, can be found at once by revolving one of its triangular faces into the plane of the picture. Each semi-ellipse becomes a semicircle, on the circumference of which is found the station point in its revolved position at D, and D A and D B are the length of two of the edges.

Fig. 28.

167. Fig. 32 exhibits the curious geometrical relations that result from the application of this process to all three faces at once. It will be noticed that the two semicircles that start from each vanishing point meet and intersect on the opposite horizon, just at the point where the perpendicular

Fig. 32.

Geometrical relations.

drawn from the vanishing point in question through the centre, V^C strikes it. If now, from each vanishing point as a centre an arc be drawn with a radius equal to the distance of that vanishing point from the station point, each arc will be the *locus* of its point of distance, and the intersection of these arcs with the horizons will give the six points of distance sought. Moreover, not only will each of these arcs pass through two out of the three revolved positions of S, but its points of intersection with the other two arcs will lie in the perpendiculars let fall from the other two vanishing points upon the opposite traces.

168. This last observation gives the means of determining all six points of distance by revolving into the plane of the picture only a single one of the faces of the pyramid, as is illustrated in Fig. 33. If the triangle V^M S V^O, right-angled at S, is revolved around V^M V^O, S will fall at D, and the points of distance D^M and D^O, two of each, are easily determined, V^M D^M being equal to V^M D, and V^O D^O to V^O D. But the *locus* of D^L passes through the point where the arc D^M D D^M cuts V^O V^C, and also through the point where D^O D D^O cuts $V^M V^C$. V^L D^L, then, is easily determined, and the two points D^L ascertained without further labor.

To find the six Points of Distance.

The several points of distance in Fig. 27, to which Fig. 33 is similar, are obtained in this way.

Fig. 33.

D, which is D^R, since D V^R is obviously equal to S V^R, enables us to determine another D^R just below V^O.

169. The phenomena of intersecting planes, with the vanishing points of their lines of intersection at the intersection of their horizons, are the same in Three-Point as in Two-Point or in One-Point Perspective, and are again and again illustrated in the plate.

CHAPTER IX.

THE PERSPECTIVE OF SHADOWS.

170. THE rays of the sun, being practically parallel, constitute a single system of parallel lines, with the same two vanishing points, 180° apart (5). Both these vanishing points may be found by looking in the direction followed by the rays. If one looks up in the direction of the rays, he of course sees the sun itself. If he looks down, away from the sun, he sees the shadow of his own head. Of the two vanishing points of the system, then, one is in the sun itself, and the other, just opposite, is in the shadow of the spectator's head, and is, of course, as far below the horizon as the sun is above it.

The phenomena of shadows.

It sometimes happens in photographs that the shadow of the camera is seen in the foreground, at the vanishing point of shadows.

171. If the sun is in front of the spectator, it is the first of these vanishing points, that in the sun itself, which is behind the plane of the picture, as in Fig. 34, and the vanishing point of the sun's rays, V^S, is above the horizon. If the sun is behind the spectator, as in Fig. 35, the other vanishing point is in the plane of the picture,

The position of the sun.

Fig. 34.

Fig. 35.

and V^S, which we now call the vanishing point of shadows, since every point throws its shadow towards it, is below the horizon. It appears in the figure in the extreme right-hand lower corner of the plate, beyond Fig. 36. If the sun is just in the plane of the picture, neither in front of the spectator nor behind him, but on one side and above, the light falls parallel with the plane of the picture, and both vanishing points are at an infinite distance upon that plane. This is illustrated in Fig. 36.

Fig. 36.

The shadow of a point.

172. The shadow of every point is accordingly a line proceeding from that point, through the air, towards the vanishing point of shadows. This line is generally invisible, the air being generally transparent; but when the air is loaded with dust or moisture this line of shadow becomes visible, as is often witnessed at sunset, when the shadows of clouds near the western horizon are thrown across the sky in parallel lines,—lines which seem to converge towards the sun in the west, and in the east to converge towards the vanishing point of shadows opposite the sun. If this invisible shadow of a point strikes any solid object, it becomes visible as a point of shadow on its surface. The *invisible* shadow of a point, then, is a line in space; the *visible* shadow of a point is a point situated where the line of invisible shadow pierces any intercepting surface upon which the shadow may fall.

The shadow of a line.

173. In like manner, the *invisible* shadow of a line is a surface in space, and the *visible* shadow of the line is a line, being the line in which this

surface intersects the surface upon which the shadow falls. If the line that casts the shadow is curved, the invisible shadow is cylindrical; if it is a straight line that casts the shadow, the shadow in space is a plane; and if the surface that receives it is also a plane, the line of visible shadow is a straight line, being the line of intersection of two planes.

In Fig. 37 both the visible and the invisible shadows of a line are represented. The sun is supposed to be behind the spectator, in such a position that the shadow of the spectator's head is thrown upon the ground within the limits of the picture at V^S. The rays of light and the shadow of every point in the line are directed towards this point. The shadow in space is seen to be a plane, and the shadow on the ground and steps is seen to be the intersection of this plane with the several planes which it encounters.

Fig. 37.

The shadow of a solid body.

174. If a solid body casts a shadow, the *invisible* shadow, passing downward through the air away from the sun, is a solid cylinder, or solid prism, according as the line upon the body that casts the shadow is a curved line or rectilinear. This line is obviously the line upon the surface of the body which separates the side towards the sun, which is in light, from the shady side. This line is called the *dividing line of light and shade.* The *visible* shadow of the solid object, seen upon any other object, is a surface, the shape of which is determined by the shadow cast by the line of light and shade.

To find the shadow of a solid body, then, is the same

thing as to find the shadow of a line, namely, the shadow of its dividing line of light and shade.

175. In finding the shadow of a point, also, the only practicable way is first to find the shadow of some line that passes through the point, and then to find in this line of shadow the point of shadow that corresponds to it. This point is easily found by drawing a line, representing the invisible shadow of the point, through the air, from the perspective of the point to the vanishing point of shadows, V^S. Its point of intersection with the visible shadow of the auxiliary line is the shadow of the point in question.

Fig. 37.

Thus in Fig. 37 the point A has its shadow at *a;* and, conversely, the shadow at *b*, at the bottom of the steps, is cast by the point B. This shows just how much of the stick throws its shadow on the ground.

Fig. 35.

In Fig. 35 the shadows of all the principal points, such as the top of the sign-post, or of the peak of the gable, are found by drawing lines to V^S, and marking their points of intersection with the shadows of the lines in which these principal points lie.

The visible shadow of a line: the intersection of the plane of its invisible shadow with the plane on which it falls.

176. The whole problem of shadows thus resolves itself into the problem of finding the shadow of a line; and as in this paper we shall consider only the case of straight lines throwing their shadows upon plane surfaces, we have to do only with rectilinear shadows, lying where the plane of the invisible shadow of the line

cuts the plane of the surface that receives it. The whole question becomes, then, simply a question of the intersection of planes.

177. Now, the line of intersection of two planes, as we have seen in the case of two intersecting roof planes, has its vanishing point at the point of intersection of the horizons of those planes (34). Hence the (visible) shadow of a line upon a given plane will have its vanishing point at the intersection of the horizon of that plane with the horizon of the plane of the (invisible) shadow of the line. And since, if any plane is given in perspective, its horizon is already known, the only thing that remains to be done is to find the horizon of the plane of the shadow. The direction of the line and the direction of the light are, of course, also given; that is to say, their vanishing points also are known.

The horizon of the plane of the invisible shadow.

178. But these two vanishing points being known, the horizon of the invisible plane of the shadow is easy to ascertain. For the horizon of a plane passes through the vanishing points of any two lines that lie in it (13, II.). Now, as may be seen in Fig. 37, the line that casts it lies in the plane of the shadow, and so does the invisible shadow of any point in that line. The horizon of the plane of the shadow accordingly passes through the vanishing point of the line that casts it, and through the vanishing point of shadows V^S, and may be found at once by drawing a line through them.

Fig. 37.

179. Thus in Fig. 34 the horizon of the shadows of the right-hand horizontal lines R, whose vanishing point

is V^R, is the line H S R, the horizon of the shadow of R, joining V^R and V^S. In the same way, if we call the plane of the shadow of L, S L; that of M, S M; that of Z, S Z, etc., we shall have the horizons of these planes, H S L, H S M, H S Z, etc., connecting V^S with V^L, V^M, V^Z, etc., respectively. As V^Z, the vanishing point of vertical lines, is at an infinite distance in the zenith, H S Z, like H R Z and H L Z, is a vertical line.

Fig. 34. The sun in front of the spectator, and behind the picture.

It is not in general very easy to follow these invisible planes of shadow in imagination, and to understand just how they go, by a mere inspection of the figure. But in the case of a vertical line, such as that of the nearest corner of the building, one can see that the plane of the shadow must be a vertical plane nearly parallel with the right-hand side of the house, but not quite so, being at a less angle with the plane of the picture. It seems reasonable, then, to find its horizon H S Z parallel with H R Z and a little further to the right.

180. Since all these planes of shadow have one element parallel to the light, all their horizons, as is seen both in Fig. 34 and in Fig. 35, pass through V^S. This point thus furnishes an illustration of the proposition (13, *b*) that the horizons of all the systems of planes that can be passed through a line, or drawn parallel to it, in any direction, pass through the vanishing point of the system to which the line belongs, and intersect each other at that point.

Fig. 35. The sun behind the spectator.

V^S, accordingly, resembles the centre of a wheel, the

spokes of which are drawn through the vanishing points of all the lines in the picture.

181. The intersection of the horizons of these planes of shadow with the horizons of the different planes on which the shadows fall gives the vanishing points of the different lines of visible shadow. Thus in Fig. 35 the horizon of the shadow of the sign-post is H S Z; and the successive portions of its shadow which fall upon the ground, R L, upon the side of the house, L Z, and upon the roof, L M, are directed to the points of intersection of H S Z with H R L, or the Horizon, with H L Z and with H L M respectively.

The vanishing point of the line of the visible shadow.

Fig. 34.

Fig. 38 exhibits these relations in a diagram. The vanishing points of shadows are marked in this plate by four letters, thus, $V^{SZ\text{-}LM}$, which signifies the vanishing point of the shadow of vertical lines falling upon the plane L M. It hardly needs to be pointed out that all the different shadows cast by the lines of any system, on whatever plane they fall, have their vanishing points on the horizon of the shadow of that system; and that all the shadows that fall on a plane, whatever kind of line casts them, have their vanishing points in the horizon of that plane.

Fig. 38.

182. The direction of the lines of shadow being thus predetermined by the determination of their vanishing points, and their length being fixed either by the limits of the plane on which they fall or by the limits of the length of the lines that cast

The initial point of a shadow.

them, everything is known about them except their exact position. To fix this it is necessary to know the position of some one point in the line of shadow. This is generally given in the conditions of the problem. In

Fig. 35.

Fig. 35, for example, so much of the shadow of the sign-post as falls on the ground is determined in position by its initial point. The shadow begins where the pole touches the ground. Thence it goes off in the direction of its vanishing point, at the intersection of H S Z with the horizon, as far as the ground extends; that is, to the wall of the house. The terminal point of the shadow on the ground is the initial point of the portion that runs up the wall, and so on.

Figs. 34, 35.

All the shadows in Fig. 34 and Fig. 35 are drawn in this way, and illustrate these principles. It is not worth while to take space to describe in detail what may now easily be understood from an inspection of the plate.

183. If no convenient spot to begin at is furnished by the conditions, it is necessary either to prolong the line and extend the plane until they meet, in which case the initial point of the shadow is the point at which the line pierces the plane, or to pass an auxiliary line, in any direction that is most convenient, through some point in the given line, and to find the point where it pierces the given plane. This point will be the initial point of the shadow of the auxiliary line; the shadow of the point selected can then be determined upon it, and the shadow of the given line drawn through that

point of shadow. The auxiliary line must always be employed when the line that casts the shadow is parallel, or nearly parallel, to the plane on which it falls.

It is generally most convenient to take this auxiliary line in a vertical direction. This is done in the case of the balloon, shown in Fig. 35. Its position being known, a line can be dropped from it upon the plane beneath, and the shadow of the balloon drawn, at the end of the shadow of the line.

184. Fig. 36 illustrates the case in which the sun is neither behind the spectator nor behind the picture, but just in the plane of the picture, throwing his rays parallel to it and to the plane of measures. This is by far the most convenient position for the sun when the objects represented are drawn in angular or two-point perspective, as they generally are. It is almost sure to produce a picturesque disposition of light and shade.

Fig 36. The sun in the plane of the picture.

It is also much simpler and easier to work than either of the other cases. For since the vanishing point of shadows is at an infinite distance, V^S is entirely off the paper, and the rays of light cross the paper at their real inclination with the ground; and not only the lines of invisible shadow, but the horizons of the planes of shadow, have the same inclination. There is an apparent difficulty in the case of vertical lines, and of other lines parallel to the picture, since their vanishing points, as well as the vanishing point of shadows, are at an

infinite distance, and it is impracticable to find the horizon of their shadows by drawing a line from one infinitely distant point to another. But it is obvious that these lines must cast their shadows in planes parallel to the picture. The shadow of such a line on any plane, then, will be parallel to the trace of that plane (80).

Fig. 36 furnishes abundant illustration of this case.

The dividing line of light and shade.

185. It is not always quite obvious, from mere inspection of a drawing, which of the edges of a solid object really determine the form of its shadow; which of its lines go to make up the dividing line of light and shade (174); which of its surfaces, that is, are turned towards the sun, and which are turned away from it. It is not easy to tell, for example, whether the farther slope of a roof is in the light or not; whether the eaves or the ridge is casting a shadow on the ground beyond. Conversely, it is not always easy to judge just where the sun must be put in order to produce the distribution of light and shade upon the different surfaces that is desired.

Sunset.

These difficulties disappear, however, if we consider that what we want to know is whether or not the sun has set, so to speak, to the plane in question, and apply to that plane the same test that we apply to the horizontal plane of the ground. If the sun is above the horizon, or the vanishing point of shadows, opposite the sun, is below the horizon, we know that the ground is in light, and *vice versa*. So of every other plane: if the sun is above its horizon, it is in light; if the sun

has set to it, and the vanishing point of shadows is above its horizon, the plane is in shade.

In Fig. 35, for example, V^s is beyond H R Z; it is above the horizon of the plane R Z, the right-hand side of the house. This side of the house is accordingly in the shade; the sun has set to it. If V^s were moved to the other side of H R Z, below this horizon, that side of the house would obviously be in the light. So of L M', the plane of the back of the roof. V^s is above H L M'; the sun has set to that plane also, and the dividing line of light and shade runs along the ridge; it is the ridge, not the eaves beyond, that casts a shadow.

Fig 35

186. But it is to be noticed that when the sun set to the end of the house which is in sight it rose to the other end of the house which is parallel to it, and as both these planes have H R Z for their horizon we must discriminate between them.

This we can do if we recall the distinction already pointed out between the surfaces that are in sight and those that are not: "A plane surface upon a solid object cannot be seen unless it is on the side of the object next the horizon of the plane" (12); that is to say, unless it is below its horizon (38).

187. Bearing this in mind, we have the following rule for the illumination of surfaces by the sun: —

A plane surface that is in sight, being turned towards its horizon, is in the light if the sun is on the farther side of its horizon, or if the vanishing point of shadows is on the hither side.

A plane surface that is out of sight is in the light if the sun is on the hither side of its horizon, or the vanishing point of shadows on the farther side.

188. It has not seemed worth while to encumber the figures with constructive lines. It is for the most part left to the intelligence of the reader to trace, point by point, the application of these principles in the various cases they present. In Fig. 35, however, a notation has been used for the outline of some of the principal shadows which will serve both to recall the principle of their construction and to indicate the point to which they are directed. The expression "S Z on R L," for instance, indicates that the outline to which it is attached is the shadow of a vertical line, Z, on a horizontal plane, R L; "S N on L M," in like manner, when applied to the shadow upon the upper roof of the iron rod which supports the chimney, signifies the shadow of a line N upon the plane L M. In both cases the line of shadow is a line of intersection of two planes, and has its vanishing point at the intersection of their horizons; in the former case at $V^{SZ.\ RL}$, where H S Z intersects with H R L, in the latter case at $V^{SN.\ LM}$, where H S N meets H L M.

Notation.

Fig. 35.

189. In a few cases the lines of invisible shadow have been indicated, converging to V^{S}, to show their use in determining the length of the visible shadow. In Fig. 34, where this is done, it will be noticed that the dotted lines drawn from the top of the posts converge at the sun, while their shadows converge

Fig 34.

to the point on the horizon below the sun. In Fig. 36 the visible shadows of vertical lines are parallel and horizontal, while the dotted lines that indicate the invisible shadows follow the real direction of the light, falling parallel to the picture, and are parallel to each other, and also to all the horizons of shadows drawn through the various vanishing points.

Fig. 36.

190. It will be observed that wherever a line is parallel to the plane on which it casts its shadow it is an element of both systems of planes; the horizons of both planes accordingly pass through its vanishing point (13, *b*), which is their point of intersection, and the shadow is parallel to the line that casts it, as it should be, having the same vanishing point. This is illustrated in Fig. 35 by the shadows cast by vertical lines upon the vertical planes, by horizontal lines upon the ground, and by the inclined lines M upon the inclined planes L M.

Shadows on planes parallel to the lines that cast them.

The shadows cast by artificial light are discussed in Chapter XIV., where will also be found, in section 313, an alternative method of finding certain shadows cast by sunlight.

CHAPTER X.

THE PERSPECTIVE OF REFLECTIONS.

191. Let us now consider how things look in a mirror, — whether in an artificial mirror, or looking-glass, or in the natural mirror formed by the surface of still water. The question is obviously a little more complicated than those we have been discussing, inasmuch as a new element is introduced. We have now to consider not only the position of the spectator and the position of the picture, and their relation to the position of the object reflected and to the mirror that reflects it, but also the relation of the object reflected and of the reflecting surface to each other. Either of these may be parallel, perpendicular, or inclined to the others.

Given, the position of the spectator, that of the picture, that of the object, and that of the mirror, it is required to depict not only the lines and surfaces of the object itself, with all their vanishing points and horizons, but the reflection of the object in the mirror, and the vanishing points and horizons of the reflection.

192. In reflections, however, as in shadows, and as everywhere in perspective, the various problems of the point, the line, and the surface are all comprised in the problem of the line (205, 206). How to draw a given

line through a given point is the only question. For the perspective of a point can be got only by finding the perspective of a line, or of two lines, passing through it, and a surface is drawn in perspective by drawing the perspective of the lines that enclose it.

The problem of reflections, then, is this: to draw the reflection of a given line in a given mirror. The position of the centre of the picture, V^C, is, of course, known, and the distance of the eye at the station point, S, in front of it; and the vanishing point of the line and the horizon or horizon of the plane of the mirror; with the position of some point in the line and of some point in the mirror.

193. Let us first take the most general case, — that of a line inclined to the picture, at any angle, taken at random, reflected in a mirror which is inclined to the plane of the picture and to the horizontal plane, in any position, taken at random.

The general case: the reflection of an obliquely inclined line in an obliquely inclined mirror.

This disposition is shown in Figs. 39, 40, and 41, which illustrate three successive steps in the solution of the problem.

Plate IX., Figs. 39, 40, 41.

In each of these is shown two sides of a room, making an angle with the plane of the picture, on one of which a mirror hangs at an angle with the wall. The plane of this mirror we will call, in pursuance of the system of notation adopted in these papers, the plane $L\dot{M}'$, since its horizontal element is obviously parallel to the edge of the floor, whose vanishing point is V^L, and its line of steep-

est slope descends to the right, in the general direction which we have called M′. $V^{M'}$ will be below V^R, and H L M′, the horizon of the plane of the mirror, will pass through $V^{M'}$ and V^L, as shown. The position of the mirror is fixed by that of its lower edge where it intersects the plane of the floor.

The line whose reflection in this mirror we are to find slants upwards to the left nearly in the direction we have called N; but as it is not parallel to the plane L Z, we will call it O, its direction being given by its vanishing point V^O. Its position is fixed by that of its nearest point, whose distance above a point on the floor is shown.

Let us also call the direction of lines at right angles to the mirror — that is to say, perpendicular or *normal* to it — by the letter T, *norma* being Latin for T-square. They will be parallel to the *axis* of the mirror, and their vanishing point will be V^T.

194. It is obvious from the inspection of either figure that O′, the reflection or *image* of the line, will look like just such another line as far behind the surface of the mirror as the line itself, or object, is in front of it; and that a line drawn from any point in the object to the corresponding point in the image — that is to say, from any point to the reflection of that point — will be normal, or perpendicular, to the mirror, and will have its vanishing point at V^T.

All these normal lines together, moreover, make up a normal plane, also at right angles to the mirror, which may be called the plane O T. And as the horizon of any

plane passes through the vanishing points of all the elements of the plane, H O T, the horizon of the normal plane O T, passes through V^O, the vanishing point of the given line, and through V^T, the vanishing point of normals.

And as, conversely, every line that lies in a plane has its vanishing point in the horizon of the plane, O′, the image of O, must have its vanishing point $V^{O'}$ also in the horizon H O T.

Finally, it is plain that the line I, in which this normal plane intersects the plane of the mirror, seems to lie equidistant between the given line O and its image O′, and to bisect the angle they make with each other; and that since it lies at the intersection of these two planes, its vanishing point must be at V^I, the intersection of their horizon.

195. The problem of reflections is solved when the line I is fixed and the point V^T; for the image or reflection of every point in the given line lies on a normal passed through the point, at a distance beyond the line I equal to that of the point itself on the hither side of it. This equal distance can be obtained, as shown, by means of the method of triangles, using a line of measures, *o t*, drawn parallel to H O T, and an auxiliary vanishing point, at V, as a point of proportional measures. By taking *o t* of any convenient length, and then taking *t o′* equal to it, the initial point of O′ upon the line T is easily ascertained. But for a complete solution of the problem, it is necessary to determine also $V^{O'}$, the vanishing point of the image.

Fig. 39 shows how V^T, and, consequently, V^I, are determined; while Fig. 40 and Fig. 41 show how I itself and $V^{O'}$ are obtained.

Fig. 39. Figs. 40, 41.

196. The first problem is how to find V^T, H L M being given; that is to say, given a plane or system of planes by its horizon, to find the vanishing point of the axis of the system, *i. e.*, its vanishing point of normals.

The solution of this problem depends upon the principle already illustrated in Fig. 29, Plate VII., that "if a line is normal to a plane its projection upon a second plane intersecting the first is perpendicular to the line of intersection." For if (Fig. 39) we pass through the station point, S, in the air, at a distance in front of V^C, equal to the line $V^C S_1$, a plane parallel to the mirror, it will intersect the plane of the picture in the line H L M′, and if we pass through the same point a line normal to that plane and parallel to the lines normal to the mirror, it will pierce the plane of the picture at V^T; for if from the station point one looks in a direction parallel to the lines of any system he will see the vanishing point of that system. Now, since the projection of the station point on the plane of the picture is at V^C, the projection of this line will be $V^C V^T$, and this line will be perpendicular to H L M′ at the point *a*. *A line, then, drawn through V^C perpendicular to the horizon of any given plane, will pass through the vanishing point of lines normal to that plane.*

Plate IX., Fig. 39.

The vanishing point of the lines normal to the mirror.

197. If now a line be drawn from the station point

S, to the point a, it will be at right angles to the line drawn from the station point to V^T. a S V^T is accordingly a right-angled triangle of which a V^T is the hypothenuse, and if this triangle be revolved into the plane of the picture about a V^T, S will fall at S^2 (V^C S^2 being taken equal to V^C S_1), and a line drawn at right angles with a S_2 will give V^T at its intersection with a V^C prolonged.

198. V^T being thus ascertained, H O T, the horizon of the normal plane, is drawn through V^T and V^O, and V^I is obtained at its intersection with H L M′ (194). **The horizon of the normal plane.**

199. Fig. 40 shows how the line I, at the intersection of the normal plane with the mirror, is determined in position; its vanishing point V^I being already found, it is necessary only to determine one point in the line I, upon the surface of the mirror. The line can then be drawn through this point and the point V^I. **Fig. 40. The point where the given line pierces the mirror.**

But it is plain, from an inspection of Fig. 39, that if the line O were prolonged until it touched the mirror its reflection O′ and the line I lying between them would be prolonged also, and that all three lines would meet at the point where O pierced the surface. This point, then, would be a point of the line I such as we are seeking.

The problem resolves itself, then, into that of finding where a line pierces a plane, both being given in perspective.

200. To find where in the figure (Fig. 40) the line O pierces the plane of the mirror L M′ we pass a vertical plane, O Z, through the line O; its horizon is H O Z. The dotted line in which this plane intersects the floor has its vanishing point at the intersection of the Horizon H R L, the horizon of the plane of the floor, with H O Z, the horizon of this vertical plane (13, III.). The point *e* is a point common to the plane L M′ and to the plane O Z; that is to say, it is one point of the line in which the vertical plane through O intersects the surface of the mirror, the point *f*, where the traces of these planes intersect, being the vanishing point of this line of intersection. The line *f e* prolonged is then this line of intersection itself, and the point *g*, where the line O prolonged meets the line *f e*, is the point where it pierces the mirror.

201. But the point *g*, being also a point of the line I prolonged, a line drawn through *g* and the vanishing point V^I gives the indefinite line I which we are seeking. Normals drawn through the extremities of the line O to V^T cut off from the indefinite line I the finite portion required. This is shown in Fig. 41.

202. Fig. 41 also shows how $V^{O\prime}$, the vanishing point of the reflection of a given line, may be obtained, V^T, V^I, and I having already been determined, and O and V^O being given.

Fig. 41. The Vanishing Point of the reflection.

It is plain that O, I, O′, and T, in Fig. 39, are all in the same normal plane, and that if from the position of the eye at the station point, S, in the air, in front of V^C,

lines are drawn to the vanishing points V^O, V^I, $V^{O'}$, and V^T, these lines will lie in the plane which, passing through the eye, intersects the plane of the picture in the horizon H O T. The lines S V^T, S V^O, S V^I, and S $V^{O'}$ will all lie in the same plane, and will be parallel respectively to T, O, I, and O′, and will make the same angles one with another. But since I bisects the angle made by O and O′ (194) so must S V^I bisect the angle at S, made by S V^O and S $V^{O}{}'$. If, then, in Fig. 41 we revolve the plane triangle V^T, S, V^I, right-angled at S, around its hypothenuse, into the plane of the picture, the horizon H O T and all the vanishing points upon it will remain where they are, and S will fall at S_4. For the real distance from the station point to the horizon H O T at b is $b\,S_3$, the hypothenuse of the right-angled triangle of which $V^C b$ is the base and $V^C S_3 = V^C S = V^C S_1$, the altitude. This distance $b\,S^3$ laid off upon a perpendicular drawn through V^C to the horizon H O T at b, gives the point S_4.

203. From this point lines drawn to the several vanishing points on H O T make the same angles one with another as do the lines O, I, O′, and T, and since I bisects the angle between O and O′, $V^{O'}$ is easily ascertained by drawing a line on one side of $S_4\,V^I$ at the same angle that $S_4\,V^O$ already makes upon the other side of it.

204. In these figures the mirror stands at an angle with the plane of the picture, and also at an angle with the ground. The case is somewhat simpler when the latter angle is 90°, the

The mirror vertical, but at an angle with the picture.

mirror being upright. The horizon of the mirror becomes vertical, and the normal lines are horizontal. If in Figs. 39, 40, and 41, we imagine the mirror set back into a vertical position, it is clear that V^T will move so as to coincide with V^R, S_2 with S_1, and H L M′ with H L Z; *f* will be at an infinite distance, and the line *e g* will be vertical. But the essential conditions of the problem will remain unchanged, and the horizon of the normal plane, the point where the given line pierces the mirror, and the vanishing point of the image will be obtained as above.

205. If the object reflected is a plane surface, its reflection must be found by finding the images of the lines that bound it, and their vanishing points. These vanishing points will of course lie in a straight line, the horizon of the reflection of the plane; this horizon can be drawn as soon as the vanishing points of any two of the elements of the plane are ascertained. It is convenient to take one of these elements parallel to the mirror (207).

The reflection of a plane figure.

206. To obtain the reflection of a point, a line must be passed through it, and its image obtained as above. The reflection of every point in this auxiliary line, including the point in question, is then easily found. But it simplifies the problem, as we shall presently see, to take the auxiliary line either parallel to the mirror or perpendicular to it (209).

The reflection of a point.

207. For the reflections of lines which are parallel to

the mirror, being in every part as far behind the mirror as the lines themselves are in front of it, are also parallel to the mirror and to the line I, in which the normal plane in which they lie intersects its surface. They are accordingly parallel to their originals, and have the same vanishing points.

Lines parallel to the mirror.

It follows that if a plane is parallel to a mirror, all the lines in it retain their vanishing points, so to speak, in the reflection, and the plane retains its horizon, *i. e.*, the horizon of the reflection is the same as the horizon of the plane itself.

208. The images of lines perpendicular to the mirror also seem to retain their vanishing point, which is the vanishing point of normals, V^T. But it is more exact to say that the *other* vanishing point, 180° away, is reflected so that its image coincides with V^T. It follows that the reflections of planes perpendicular to a mirror have the same horizons as the planes themselves; for the vanishing points of the parallel elements and of the normal elements are alike unchanged.

Lines perpendicular to the mirror.

209. It appears, then, that *the reflections of lines and planes parallel or perpendicular to the reflecting surface have the same vanishing points and horizons as their originals,* whatever the position of the reflecting surface.

210. The general problem of the reflections of lines and planes, whether parallel, perpendicular, or inclined to the mirror, having thus been discussed, it only re-

mains to consider the special cases in which the mirror itself is parallel or perpendicular to the picture. In both these cases the problem is a very simple one.

The mirror perpendicular to the plane of the picture.

211. *When the mirror is perpendicular to the plane of the picture,* as in Figs. 42 and 43, the normals are parallel to the picture, their vanishing point is at an infinite distance, and the horizon of the normal plane is at right angles to that of the mirror. The horizon of the plane of the mirror passes through the centre, V^c; the plane drawn through the station point, S, intersecting the picture in this horizon, is perpendicular to the picture, and lines drawn from the station point to the vanishing point of a line and to that of its reflection strike the picture at equal distances from this horizon.

The vanishing point of the image of a line, then, inclined to the face of a mirror which is perpendicular to the picture, *is as far on one side of the horizon of the mirror as the original vanishing point is on the other.* on a line at right angles to the horizon.

If a line is parallel to the picture, so that its vanishing point is at an infinite distance, the image is inclined to the horizon of the plane of the mirror at an equal angle on the other side.

212. If a plane is inclined to a mirror that is perpendicular to the picture, one element of the plane will nevertheless lie parallel to it, and that element will retain its vanishing point, which will lie in the horizon of the plane of the mirror (209); every other element will, so to speak, shift its vanishing point to the other side

of the horizon of the reflecting plane (211). *The horizon of the image will then cross the horizon of the mirror at the same point with that of its original,* making equal angles on the other side.

213. *When the mirror is parallel to the plane of the picture,* having the vanishing point of its axis at the centre, V^C, a line, or system of lines, inclined to the mirror has the vanishing point of its image *as far from the centre V^C, in one direction as that of the line or system is in the other,* on a line drawn through the centre. If a line goes up and to the right, its reflection will of course seem to go down and to the left at equal angles.

The mirror parallel to the plane of the picture.

214. It follows that if a plane is inclined to a mirror set parallel to the picture the horizon of its image *is parallel to that of the plane itself,* on the opposite side of the centre, V^C, and equidistant from the centre.

215. These points are illustrated in Fig. 42. The spokes of the spinning wheel are parallel to the right-hand mirror, the axle is perpendicular to it. All retain their vanishing points.

Fig. 42.

The box on the left, with its cover, presents four systems of lines, two horizontal, L and R; two inclined, N and O. The reflection in the floor retains V^L and V^R, but exchanges V^N and V^O for V^{N1} and V^{O1}, on the opposite side of the Horizon (209, 211).

The reflection of the box in the second mirror, on the left, has for vanishing points V^{L2}, V^{R2}, V^{O2}, V^{N2}, across

the horizon of the mirror H X A, the plane $L Z_2$ having $H L Z_2$ for its trace, inclined to H X A equally with H L Z (212).

The third mirror, parallel with the picture, in like manner gives V^{L3}, V^{R3}, V^{N3}, and V^{O3}, for the vanishing points of the main lines of the reflection of box (213).

216. The reflections are themselves reflected just like their originals, as in mirror No. 2. $V^{N1.2}$ is the vanishing point of the lid of the box, reflected first in the polished floor and then in the inclined mirror.

217. Fig. 43 illustrates these principles by the phenomena of reflections in water, a mirror perpendicular to the plane of the picture.

Fig. 43. Reflections in water.

The steps have their vanishing point at V^M and $V^{M'}$, their reflections at $V^{M'}$ and V^M, respectively.

218. It is to be observed that the phenomena of reflection enable us to determine the real distance of isolated objects, such as birds or distant mountains, the point on the plane of the water directly below them being midway between the object and the image. The distance of this point is the horizontal distance of the object.

It is also to be noticed that the different size and inclination of the sticks in the foreground, which are not very obvious in themselves, are made conspicuous by the difference of their reflections, as it would be indeed on solid ground by the difference of their shadows. These objects exhibit very clearly the relation of a line

and its reflection to the normal plane in which they lie, and to the line in which this plane intersects the mirror.

That the horizontal line of birds should be reflected in an inclined line, and the inclined line of birds in a horizontal line, is easily understood by observing the line of dots on the surface of the water, midway between the birds and their reflections.

Each bird, as well as each mountain top, and indeed every other point, has its reflection as far below the surface of the water directly beneath it as the point itself is above. The plane of the surface of the water is supposed to extend, of course, beneath the land.

CHAPTER XI.

THE PERSPECTIVE OF CIRCLES.

IN the ten preceding chapters we have considered all the principal problems of plane perspective embraced in our scheme. That is to say, we have shown how to obtain, upon a plane surface, the perspective representation of a straight line; whatever the position of the surface, and whatever the position of the spectator, we have shown how to obtain the position, magnitude, and direction of the representation of a line, when the position, magnitude, and direction of the line itself are known. The problems of shadows and of reflections have also been fully discussed, so far as concerns plane surfaces.. Throughout the whole investigation it has been shown that the problem of the line includes the problems of plane and solid figures and of the point. In every case the vanishing point of every line and the vanishing line, or horizon, of every plane has been ascertained, the solution being considered incomplete until this was accomplished.

219. The only lines included in this survey have accordingly been right lines, and the only plane or solid figures have been such as are bounded by right lines. Any other line or outline in perspective, as elsewhere in geometry, must in general be treated as a series of points,

the perspective representation of each point being obtained separately. But to this rule the circle, here as elsewhere, constitutes an exception, its exceptional importance making it worth while to give it special consideration, while its peculiar geometrical properties render the investigation exceptionally simple and easy.

We shall find also that the study of the circle in perspective, and of its derivatives, the cylinder and the sphere, introduces a new set of most interesting phenomena, the investigation of which will, in the two subsequent chapters, lead to theoretical and practical conclusions of the first importance.

220. The perspective representation of a circle will obviously be the line in which the plane of the picture intersects a cone of rays of which the vertex is in the eye of the spectator, at the station point, and the base is the circle itself. The theory of conic sections establishes the fact that this line of intersection will be a circle, ellipse, parabola, or hyperbola, according to the angle at which the plane of the picture cuts the cone of the rays, and this whether the axis of the cone be at right angles to the circle or inclined to it. In other words, the cross-section of the cone, perpendicular to its axis, may be either a circle or an ellipse. If the secant plane is parallel to the base, or equally inclined to the axis in a contrary direction, making what is called a *sub-contrary* section, the perspective will be similar in shape, though not of course in size, to its original.

The perspective of a circle, a conic section.

221. This is illustrated in Fig. 44, Plate X., and also
Fig. 44. by the three figures 54, 55, and 56, in Plates XI., XII., and XIII., all of which show how horizontal circles, whether below or above the eye, will appear when viewed from different positions.

In Fig. 44 the spectator is shown as regarding the circular room on the right from three different positions. At S^A he is outside the room; the plane of the picture, *p a,* cuts completely across the cones of rays, and the sections are ellipses, as shown at A, below. At S^B he is just upon the edge of the circle; the plane of the picture, *p b,* cuts the cones in a direction parallel to one side, and the sections are parabolas, as seen below at B. At S^C the spectator is fairly within the circle, and the intersection of the vertical plane, *p c,* with the cones of rays gives
Figs. 54, 55, 56. hyperbolas, as at C. Figs. 54, 55, and 56, which are reproductions of engravings of circular halls in the Vatican Palace, excellently illustrate these elliptical, parabolic, and hyperbolic lines: the first being drawn, presumably with a *camera lucida,* from a point outside the circle; the second, from a point just on the edge of the room; the third, from a point within it.

222. Fig. 44 also illustrates the case in which the cone of rays is intersected by the plane of the picture in such a way as to give a sub-contrary section. The small horizontal circle forming the *eye* of the dome is the base of a cone of rays which is cut by the plane *a p,* at an angle with the axis of the cone equal to that made by the circle itself, but taken in a contrary direction. Both

are obviously angles of 45° with the axis, which is accordingly at 45° with the horizon. The perspective of the circle is accordingly a circle (220), as is shown below. In Fig. 54, also, Plate XI., the perspective of the circle at the top of the dome is almost exactly circular.

223. Fig. 45 illustrates more fully the sub-contrary section spoken of in the previous paragraph, and shown in Fig. 44. A_1 A_1 shows the circle in its own plane, with its centre beyond the axis of the cone; B_1 B_1 shows its perspective in the plane of the picture, *p p*, with the centre of the cone of rays below its centre, and the point *a*, representing the centre of the original circle, lower still; E_1 E_1 shows the real shape of the cross-section E E, taken at right angles with the axis of the cone. This is an ellipse, whose centre coincides with the axis of the cone of visual rays, the centres of both circles being also given, one on one side and one on the other. The projections of the respective centres are shown in each case, at *a*, *b*, and *e*.

Fig. 45.

The line D D, parallel to A A, shows that a horizontal section of the cone taken at this place must be a circle like A_1 A_1; and since the sub-contrary section at B B is symmetrical with it, about the axis of the cone, it follows that B_1 B_1 also must be a circle.

224. It is to be noticed that the ellipse E_1 E_1 is the appearance that the circle would present from the station point, S. It would not appear as a circle, though its perspective is a circle. But neither does this perspective circle appear as a circle. It, too, is fore-

shortened into an ellipse in the sight of a spectator at S (230).

225. In fact, unless a circle is situated just at the centre of the picture, the ellipse which represents it in perspective is of a different shape from the ellipse which it presents to the eye. Horizontal circles, for instance, always present to the eye horizontal ellipses; ellipses, that is to say, whose major axes are horizontal. But in perspective such circles, unless just above or below the

Fig. 47.

centre, V^C, have their axes inclined, as we may see in Fig. 47. Yet these oblique ellipses, when seen from the proper position, the station point in front of V^C, are themselves apparently changed in shape by the effect of perspective, and foreshortened into horizontal ellipses.

This seeming distortion in the perspective, which makes the outline in the drawing of a different shape from the apparent outline of the thing drawn, will form the subject of the next chapter.

226. In general, of course, the station point is outside the circle to be represented, so that practically the problem of putting a circle into perspective is this: to find the ellipse which represents it.

To draw the perspective ellipse.

The simplest and generally the most efficient way to do this is to suppose a square or octagon to be circumscribed about the given circle, at any angle that may be most convenient. The centres of these sides give, of course, four or eight points of the required ellipse. As the direction of the sides gives the

direction of the ellipse at these points, it can easily be drawn with sufficient accuracy for practical purposes. If greater accuracy is required the number of sides of the circumscribing polygon can be increased.

This is illustrated in Fig. 47. Fig. 47.

227. But in order to draw an ellipse with absolute precision it is necessary to find its centre, and the direction and length of its axes or principal diameters.

It is obvious from Fig. 44 and from Fig. 54, Plate XI., that the perspective of the centre of a circle does not coincide with the middle point of the ellipse, as indeed it cannot, since the farther half of a circle must appear smaller than the nearer half, and its radii shorter. Neither do the extreme points of the ellipse represent extreme points of the circle; the highest point in the perspective of an arch, as may be seen in the arched windows and niches of Fig. 54, is the perspective, not of the highest point in the arch, but of a lower point nearer the spectator. In fact, although a circle put into perspective appears as an ellipse, the diameters of the circle do not become diameters of the ellipse, but chords, which intersect at a point situated beyond the centre. On the other hand, the diameters of the ellipse, meeting and intersecting at its centre, are the perspectives of chords of the circle, which meet and intersect at a point within the circle nearer to the spectator than its centre. The tangents at the extremities of each diameter of the circle are parallel; but their perspectives of course converge to a vanishing point; and since, for each circle, all these

Its centre and extreme points.

tangents lie in the same plane, these vanishing points all lie in the same straight line, which is the horizon of that plane. See Fig. 46.

Fig. 46.

The Pole and Polar Line.

228. These phenomena afford a curious illustration of certain well known geometrical properties of the ellipse. If a point be taken at random anywhere within a circle or an ellipse, and chords be drawn through that point, then the tangents drawn from the extremities of each chord will have their point of intersection upon a right line. This line is called a *polar line*, the point assumed being called a *pole*. As a mere geometrical proposition this seems to have no special significance; but the phenomena of perspective give it meaning. For a circle seen in perspective becomes an ellipse, its centre a pole, its diameters chords, and the polar line is the horizon upon which meet the tangents drawn from the extremities of its diameters.

To find the centre of a given ellipse.

229. Fig. 46 exhibits these relations, and illustrates also the further proposition, which, however, does not seem to admit of similar interpretation, that if lines are drawn from the points where the tangents meet through the middle of the chords they will pass through the centre of the ellipse. This property we shall find a use for presently (235, 238).

Fig. 44, A, Fig. 45.

230. Although in Fig. 44, A, and in Fig. 45, the small vertical circle is the perspective of the horizontal circle at the top of the dome, so that, when seen from the station point, the two circles seem to coincide, yet, as we have seen (223), their centres do not

coincide, and neither of them coincides with the axis of the cone. The centre of each circle appears as a pole of the other. If the cone be cut by a plane at right angles with its axis, as at E E, Fig. 45, the section will be an ellipse, E_1 E_1, of which the axis of the cone will give the centre, and of which the centre of the upper circle will be a pole situated just below the axis, and the centre of the lower circle will be another pole just above it. In the horizontal circle the centres of the ellipse and of the vertical circle become poles, and in the vertical circle the centres of the ellipse and of the horizontal circle become poles.

An ellipse, of course, may by perspective be foreshortened into a circle, — the centre of the ellipse becoming a pole, just as a circle appears like an ellipse.

231. It will be noticed in figures 54, 55, and 56, and also in Fig. 47, that the vertical circles whose centres are on the Horizon, and the horizontal circles whose centres are exactly above or below the centre of the picture, V^C, lie symmetrically on the paper; that is to say, that their principal diameters, their major and minor axes, are vertical and horizontal, but that other ellipses have their axes more or less inclined.

Another and more comprehensive statement of this phenomenon is this: that if a line drawn through the centre of a circle normal to its plane, like the axle of a wheel, crosses the centre of the picture, V^C, one axis of the ellipse that represents the circle will coincide with this line, and will also be directed towards V^C, and the

other will be at right angles to it. Other circles, which do not, as it were, thus *face* the centre, V^C, will be projected in ellipses the direction of whose axes it is more difficult to determine.

Fig. 47 shows a number of circles, three of which, A A, B B, and D D, are vertical, and accordingly appear in plan as right lines, and two, E E and F F, are horizontal. These appear in perspective at $A_1 A_1$, $B_1 B_1$, $D_1 D_1$, $E_1 E_1$, and $F_1 F_1$, respectively. In all of these except the last, one of the principal axes passes through the centre, V^C. In $F_1 F_1$, and also in the circles $A_2 A_2$ and $B_2 B_2$, the position of the principal axes is, so to speak, accidental.

Ellipses that face the centre.

Fig. 47.

232. Let us first take the case of the ellipses which represent circles that *do* face the centre, V^C, and which accordingly lie symmetrically about a normal line joining this point with the perspective of their centres. This line will, of course, also pass through the centre of the ellipse, as in $A_1 A_1$, $B_1 B_1$, and $E_1 E_1$, Fig. 47.

233. If, as in $A_1 A_1$ and $B_1 B_1$, two of the sides of the circumscribing square or octagon are parallel to the plane of the picture, and their perspectives consequently parallel to each other and perpendicular to the normal line, the line joining their middle points will be the minor axis of the ellipse ; the major axis will cross it at its middle point, and it will only remain to ascertain the length of this major axis.

234. Fig. 47 shows how this is done. Let A A, B B,

and D D, in the plan, be three parallel and similar circles, all touching the plane of the picture, and the last, as appears from the perspective below, standing edgewise to the spectator at S. If the space between them were filled up with other such circles, they would all together constitute an elliptical cylinder, the apparent vertical dimension of which would be the apparent height of each of the circles and of the major axis of the ellipses that represent them. Now let the plane containing the circle D D and the station point, S, be revolved into the plane of the picture around the vertical line H R Z, in which the two planes intersect. S will, of course, fall at the point of distance D^R; the circle D D will appear of its true shape and size; lines drawn from D^R tangent to the circle D D thus revolved will determine the highest and lowest points visible from S, and the points where these lines cut the line H R Z will show the perspective of these points in the plane of the picture, and fix the apparent height of D D. The circles B B and A A will appear to be of the same height as D D, and D_1 D_1 will be the length of the major axes of the ellipses that represent them.

235. When, as in the case of the circle E E, the tangent lines are not parallel to the picture, the square or octagon that encloses the circle being in angular perspective, instead of being in parallel perspective, as in the previous instance, the centre of the ellipse must be obtained as above explained (229) and shown in Fig. 46, by bisecting two of the chords, and drawing lines from the vanishing points of their tangents. The principal axes of the ellipse may then be drawn, one towards the centre, $V^C_,$

and the other parallel to the picture, that is to say, parallel to the horizon of the plane in which the circle lies (38). The length *e e* of the latter, or major axis, may then be found by direct projection, as in the figure.

236. To find the length of the minor axis, the major axis and one point of the ellipse being given, it is only necessary to employ the usual device, shown in Fig. 48, founded upon the proposition that if a semicircle be erected on the major axis of an ellipse the distance of the different points of the ellipse from this axis will be proportional to that of the corresponding points of the semicircle. Thus, in the figure, one point x being given on the ellipse, and x' and b' obtained on the circle, the point b at the extremity of the minor axis is easily found, since the chords $b\,x$ and $b'\,x'$ meet on the line of the major axis prolonged.

Fig. 48.

237. When the ellipse does not lie opposite the centre, V^C, as is the case with $F_1\,F_1$, and with $A_2\,A_2$ and $B_2\,B_2$, in Fig. 47, the normals drawn through the centres of these circles not passing across the centre of the picture (231), the only way to obtain the principal diameters, or axes, is first to obtain a pair of *conjugate* diameters. Conjugate diameters are diameters each of which is parallel to the tangents drawn through the extremities of the other. The axes are that pair of conjugate diameters which are at right angles with each other; and one is always the longest diameter that can be drawn in a given ellipse, and the other the shortest.

Ellipses that do not face the centre.

Fig. 47.

238. The quickest way to obtain a pair of conjugate diameters in oblique ellipses such as $A_2 A_2$, $B_2 B_2$, and $F_1 F_1$ is to construct first such horizontal and vertical ellipses as $A_1 A_1$, $B_1 B_1$, and $E_1 E_1$, respectively, opposite the centre, and to obtain their principal axes as just described. If now the centre of each oblique ellipse is found, as above (229), and lines passed through it perspectively equal and parallel to these principal diameters, they will be conjugate diameters of the oblique ellipses. They will not be at right angles, but each will be parallel to the tangents drawn at the extremities of the other; one will be parallel to the plane of the picture and the other perpendicular to it, and directed to the centre, V^C.

239. Conjugate diameters of the ellipses $A_2 A_2$, $B_2 B_2$, and $F_1 F_1$ being thus determined, their principal diameters or axes can now easily be obtained by the well known method of shadows, as in the figure.

240. Fig. 49, *a*, shows this ingenious device more in detail. It is called the Method of Shadows, because the ellipse is regarded as the shadow or projection of a circle. The process is this: —

Fig. 49, *a*.

A tangent being drawn at the extremity of one diameter parallel to its conjugate, a circle is erected also tangent at the same point, of such size that the ellipse might be its shadow. The shadow of every diameter of the circle will be a diameter of the ellipse, and the shadows of any two diameters of the circle which are at right angles with each other will be conjugate diameters

To obtain the principal axes of an ellipse when two conjugate diameters are given by the method of shadows.

of the ellipse, since the tangents at the extremity of one will be parallel to the other. The given conjugates of the ellipse are shadows of those diameters of the circle which are perpendicular to and parallel with the tangent line common to both circle and ellipse. Since the shadow of the diameter parallel to the tangent is also parallel to the tangent, the diameter and its shadow are parallel to each other and must be of the same length. This fixes the size of the circle, the distance of whose centre from the end of one diameter is equal to half the length of its conjugate.

241. It now only remains to find in this circle a pair of diameters at right angles to each other whose shadows will also be at right angles. But since it is plain that if these diameters are prolonged till they reach the tangent line their shadows will also be prolonged, and will reach the tangent line at the same points, the problem becomes a very simple one. It is only necessary to find two points on the tangent line which make right-angled triangles both with the centre of the circle and with the centre of the ellipse; that is to say, two points such that the portion of the tangent lying between them shall be the common diameter of two semicircles passing respectively through these two centres. The common centre of these semicircles must be a point on the tangent line equidistant from the two centres; a point easily found by erecting a perpendicular upon the middle of the line connecting them, as is done in the figure. Semicircles struck from this point *c* as a centre, with a radius equal to its distance from the centre of the circle or of the ellipse, give

the points *a* and *b*, through which diameters can be drawn in the circle whose shadows, drawn through the same points to the centre of the ellipse, are axes or principal diameters of the ellipse, both sets of diameters making right angles with each other.

242. As the centre of the ellipse is the shadow of the centre of the circle, the line that joins these centres may be considered to give the direction of the light, and lines drawn parallel to it through the extremities of the diameters of the circle will give the extremities of the corresponding diameters of the ellipse, or the length of the axes.

243. This operation, though long in the description, is simple in practice, and requires very few constructive lines, as is seen in Fig. 49, *b*, in which the operation just described is repeated with no more construction lines than are necessary.

Fig. 49, *b*.

This method is used, as has been said, in finding the axes of $A_2 A_2$, $B_2 B_2$, and $F_1 F_1$, and all the necessary construction lines are given in the figure.

244. When only a portion of a circle is to be put into perspective it is generally best to construct the ellipse which represents the whole circle, and then to use so much of it as may be required. This is eminently the case in sketching from nature or from the imagination, where it is difficult to determine the character of the perspective curve without aid from geometrical considerations. In drawing pointed arches, for instance, the character of the intersecting arcs

The perspective of arcs of circles, and of arches.

is best ascertained by completing the circles of which they form a part, as in Fig. 50. An inspection of the figure shows that when the arch is above the eye the nearer half is represented by the part of an ellipse at which the curvature is the most rapid, near the extremity of the major axis; and the further half by the flattest portion, near the extremity of the minor axis. When the circle is below the eye the nearer part is the flattest.

Fig. 50.

This figure also shows that when a row of circles is put into perspective their major axes are not parallel, their inclination to the horizon of the plane in which the circles lie diminishing as they approach it. This is illustrated also in Fig. 53.

Concentric circles.

245. Figs. 51, 52, and 53 show three different ways of drawing concentric circles. Since concentric circles have the same centre, the ellipses which constitute their perspectives have of course the same pole and polar line; but the ellipses have not the same centre, nor are their axes parallel.

Fig. 51.

246. The first method is shown in Fig. 51; it is applicable to the case where the perspective of a circle is obtained by means of a circumscribed square or polygon. A second circle, concentric with the first, is easily obtained by means of a concentric polygon, as shown.

Fig. 52.

247. Fig. 52 shows how the second ellipse can be found when the first has been already determined in any way. Let a line of measures be

drawn through the pole which is the perspective of the centre of the circle, parallel to the polar line, or horizon of the plane in which the circle lies. If now any chord *a a*, representing a diameter of the circle, be drawn through this pole, and lines be drawn from its extremities to any point V upon this line or horizon, it will cut the line of measures at two points, *a′* and *a′*, whose distance from the pole is the same. If now two other points, *b′* and *b′*, be taken, also equidistant from the pole, and lines be drawn through them, the points *b* and *b*, in which they intercept the same chord, will be points of an ellipse which represents a circle concentric with the given circle, and as much smaller as *b′ b′* is smaller than *a′ a′*. In the figure the radius of the smaller circle is one half the radius of the larger one, *b′ b′* being one half of *a′ a′*. It is obvious that since the lines meeting at V are parallel in space, the lines *a a* and *a′ a′* are divided proportionally. Any number of points can be obtained in the same way as *b b*.

248. A third method of putting concentric circles into perspective is shown in Fig. 53, — a figure which, like Fig. 47 in plate X, shows three equal circles, A A, B B, and D D, lying in parallel planes and equally distant from the picture, the last of which stands *edgewise* to the spectator, so that it coincides with a portion of H R Z, the horizon of the parallel planes. In the figure it is supposed that A A is the given circle, concentric with which it is required to draw another circle E E.

Fig. 53.

249. To effect this the circle D D is first found by

cutting off from H R Z a portion equal in height to A A, this height being measured above and below a line passing through the centres of the three circles. Parallel to this line let a second line be drawn through any point, 1, of the circle A A, to the corresponding point, 3, of the circle D D; and upon this line let any convenient point, as 2, between 1 and 3, be taken as the corresponding point of a third circle, B B. If now a fourth point, 4, be taken upon the line through the centres, the line 4 2 5 will be an element of a cone whose vertex is at 4 and whose base in the plane of the circle A A is a circle concentric with that circle. The intersection of the line 4 2, prolonged, with a radius of A A drawn through the point 1, fixes the point 5, in the circumference of the circle E E. By drawing other lines, parallel to the line 1 2 3, through other points of A A, any number of other points in B B and E E may now easily be obtained.

250. The radii of the circles E E and A A (or B B) are obviously proportional to the distance of the point 4 from the centres of E E and B B, and also to the chords drawn through the common centre of A A and E E parallel to D D, that is, to the trace T R Z. By changing the position of 4, the size of E E may be made larger or smaller.

251. It will be noticed, 1st, that this method not only gives the means of finding a larger circle concentric with A A, and lying in the same plane, but also of finding an equal circle, B B, lying in a parallel plane; 2d, that it makes no difference in what direc-

tion the axis of the cone passing through the three centres is originally drawn, provided it is parallel to the plane of the picture; and 3d, that this method is as serviceable in drawing a concentric circle smaller than the given circle as in drawing a larger one; for if E E were the given circle a reversed process would give A A.

252. As to the figures 54, 55, and 56, in Plates XI., XII., and XIII., it is perhaps worth while to say that the excessive distortion apparent in them is due simply to the fact that the station point, or proper position of the spectator, is in each of them within three or four inches of the page. This is within the limits of distinct vision. But by looking through a pin-hole in a card the prints can be distinctly seen when held even at the end of one's nose; and when so viewed it will be seen that not only the ellipses in Fig. 54, but the parabolas and hyperbolas in Figs. 55 and 56, look like circles, as they should. The apparent distortion entirely disappears.

Figs. 54, 55, 56.

CHAPTER XII.

DISTORTIONS AND CORRECTIONS. — THE HUMAN FIGURE.

253. It has been pointed out in the previous chapter that the perspectives of circles often look very queer; the ellipses by which they are represented seem unaccountably and even unnaturally inclined, their principal axes slanting in directions difficult to anticipate. The effect of this is particularly objectionable when the circle forms the base of a cylinder or when it is horizontal. The base of a cylinder always presents the appearance of an ellipse whose major axis is at right angles with the axis of the cylinder, and it is offensive to find it drawn otherwise, as in perspective often happens.

Plate XIV.

254. A horizontal circle always appears to the eye as a horizontal ellipse, as an ellipse, that is to say, whose major axis is parallel to the horizon and whose minor axis is perpendicular to it, and it is extremely unpleasant to see it drawn with the axes inclined. This is illustrated in Plate XIV., Fig. 58, by the perspective plan of the capital at the top of the figure and that of the base at the bottom. The effect of this would be so disagreeable if the curves of the capital and base were inclined in

Fig. 58.

Horizontal circles.

like manner, that it is customary to introduce a certain *correction*, as it is called, as is done in the figure. These lines are accordingly drawn as horizontal ellipses, just as if these objects faced the centre of the picture (231).

255. Fig. 59, *c*, still further illustrates this point, showing that in the column at the centre of the picture the ellipses are horizontal, and that the others are more and more inclined as they are farther removed from it, which looks like an unnatural distortion. In this figure, moreover, the outer columns, which as seen from the station point at S would look the smallest, since they are farthest from the eye, are on the contrary drawn larger in diameter, an apparent distortion even more offensive than the other. So with the spheres by which the columns are surmounted. The outline of a sphere always looks like a circle; it is not agreeable to find it drawn as an ellipse. But in perspective it must always be an ellipse, unless its centre is just at the centre of the picture; for the perspective representation of the sphere is the section of a right cone with a circular base, the base of the cone being that great circle of the sphere which separates the side of the sphere one can see from the further half of the sphere that he cannot see; it must always be an ellipse unless the axis of the cone is perpendicular to the plane of the picture.

Fig. 59, *c*. Distortions of circles, cylinders, and spheres.

256. Of course all these distortions disappear when the eye is at the station point, at a proper distance in front of the picture, opposite the centre V^c. From that point of view the perspective lines exactly cover and

coincide with the oulines of the objects. But practically it is impossible for the spectator always to be exactly at the station point, and since from every other point, circles, cylinders, and spheres appear, in general, to be more or less distorted in the manner we have just seen, it is customary here also to apply corrections. These corrections are palpable violations of the rules of perspective made in order to avoid the disagreeable consequences of deserting the station point. They consist in drawing all horizontal circles as horizontal ellipses, whether opposite the centre of the picture or not; in always drawing the elliptical representation of the base of a cylinder at right angles with the cylinder itself; and in drawing all spheres as circles. If a row of columns, moreover, is parallel to the picture, they are always made of the same diameter, as if seen in elevation, and if the direction of the row is slightly inclined to the picture care is taken to diminish their width a little as they recede.

Corrections.

257. Fig. 59, *a,* illustrates these corrections, and shows further how the same treatment is sometimes extended to the octagon. The right-hand side of the octagonal figure at the top is drawn steeper than it ought to be, not being directed to its proper vanishing point, in order to remedy the apparent distortion seen in the corresponding figure below.

Fig. 59, *a*.

258. Fig. 59, *a*, shows also the effect of applying to vertical circles and semicircles the same corrections as to horizontal ones. The circular window which in the figure below is drawn in true perspective as an oblique

ellipse is here shown as a vertical one, a change which will probably be regarded by most persons as an improvement. The effect of a similar correction in the semicircular-window head beyond is less happy; it makes the nearer half look much too big, and obviously throws the imposts, or points where the arch begins, quite out of level.

259. Since these so-called corrections change and generally diminish the apparent size of the circles, cylinders, and spheres to which they are applied, the relation of these objects to other objects is necessarily changed at the same time. In the first place, more of the background has to be shown than can really be seen. In the figure, for example, the openings between the columns are increased, and objects are seen beyond which in point of fact would be hidden. This discrepancy is not very important and in general would hardly be noticed, but the altered relations between these circular figures and other objects in their immediate neighborhood is a more serious matter. The square abacus between the shaft and the sphere that surmounts it looks too big for its place if left without correction, and looks smaller than its fellows if reduced as the sphere and cylinder are. So also when an octagon occurs in immediate connection with a circle. If its shape is adjusted to that of the *corrected* ellipse, its want of harmony with the rest of the drawing often becomes painfully apparent. A satisfactory adjustment may sometimes be effected by a compromise, the ellipse being made not quite horizontal and the octagon or

square being not quite harmonized with it. But a perfectly satisfactory adjustment is in some of these cases, and notably in the case of a row of columns, almost impossible.

These difficulties are of course greater as the objects in question are further removed from the centre of the picture, and may be diminished or removed altogether by so taking the position of the picture and that of the spectator that the circular object is at or near the centre V^C.

Distortion of all figures.

260. Although the distortions of circular and spherical objects are, in general, the only ones that call loudly for correction, and it is to them alone that correction is systematically applied, it is obvious that a similar distortion must exist for all objects equally distant from the centre of the picture, the so-called distortion consisting in this, that the shape of the object in the drawing is different from the shape which the real object presents to the eye (224, 225). This is in fact implied in the fundamental principle of perspective, the principle that a perspective drawing will look right from only one point, namely, the station-point. Now as from the station-point every part of the picture except the centre is viewed obliquely, *askance*, as it were, everything *must* be drawn of a different shape from what it appears in order that when the drawing is looked at thus obliquely it may appear as the object itself does when looked at directly. By the very theory of perspective only the object just opposite the eye, seen

along the axis of the picture, just at its centre, is drawn as it looks. Everything else is, so to speak, distorted. The outline given to it is not its real outline, but one which will look like its real outline when seen sideways from the position assigned to the spectator. This distortion is inevitable, and every object in a perspective drawing, except the one at the centre, is always distorted.

This is most noticeable when objects are drawn in parallel perspective, as, for example, in Fig. 23, Plate VI. But all objects are more or less distorted and exaggerated in size. They are *stretched out* in a direction away from the centre of the picture, just like the shadows in Fig. 60.

261. The disfigurement produced by this does not of course become very obvious, except for circles and spheres, so long as objects are not far removed from the centre, that is to say, so long as the picture is of moderate extent. The limit commonly assigned to a perspective drawing is sixty degrees, that is to say, the width of the picture should not be greater than its distance from the station point. But this implies that the centre V^c is in the middle of the picture, which as we have seen is often not the case, and it is better to say that no part of the picture should be distant from the point opposite the eye more than half the distance of the spectator in front of it. But even within this range the distortion even of rectilinear objects is sometimes intolerable, and great caution must always be used in regard to objects situated at the edges of the picture.

262. This limit of sixty degrees is obviously an arbitrary one, and only means that by the time it is reached the distortion begins to be noticeable. It is foolish to say, as is sometimes said, that this is fixed because sixty degrees embrace all that one can see without turning his eyes, or as others say, without turning his head, and that this is accordingly the natural range for a picture. For one has to turn his eyes, more or less, to see directly anything larger than a pin-head, held at arm's length; and he has no need to turn his head even to embrace a horizon of ninety degrees. Besides, why should not one turn his head as well as his eyes in looking at a picture as well as in looking at nature? If he is at the station-point, where he ought to be, turning his head cannot make things look wrong, and if he is not there keeping it still will not make them look right.

The extent of the range of the picture.

263. There is, nevertheless, a remarkable difference between turning the head to look at a thing, and merely turning the eye. The plane of the picture, so to speak, regarding the aspect of nature as a picture, is conceived to be parallel to one's face. Turning the head seems to alter the position of the plane of projection. If one looks straight down a street he seems to see it in parallel perspective. If, keeping his eye fixed on the same spot, he turns his head, the street seems now to be in angular perspective, though the image on the retina has not been disturbed. The right-angles will seem to become acute and obtuse. A horizontal circle on the floor or ceiling will look like an ellipse

with a horizontal axis if you face it. Turn your head away, without moving your eyes, and though it really looks just as it did before, being regarded from the same point, its axes will seem inclined.

264. What has been said of cylinders and spheres strictly applies to the human figure, which may be regarded, in a rough way, as a cylinder surmounted by a sphere. Perspective distortion is here even more intolerable than in the case of the more exact geometrical solids, and the need of correction is more imperative.

The human figure.

This is excellently illustrated by the phenomena of the familiar parlor amusement called "Chinese Shadows," in which a sheet is hung across the middle of a room and the shadows of the performers on one side are thrown upon it for the entertainment of spectators on the other side. A single lamp is used, and it is obvious that all the shadows except the one just opposite the lamp and on a level with it, must be more or less distorted as they are more or less removed from this centre. But it is also obvious that if one of the spectators places himself exactly opposite the lamp and as far in front of the screen as the light is behind it, the distorted outlines will be foreshortened into the true shape of the figures on the other side as seen from the place occupied by the flame.

Chinese Shadows.

265. An historical picture then, if painted in true perspective, with all its figures so drawn as to present their true aspect to the spectator

Historical pictures.

standing at a given point in front of it, would have all its personages as much out of drawing as are Chinese Shadows upon a screen. Fig. 60, which exhibits the results of an experiment with a group of statuary and with half a dozen round balls, illustrates these conclusions. They are, as happens in perspective to all objects (260), *stretched out*, in a direction away from the centre, V^C.

Fig. 60.

266. No such picture of course was ever painted, painters always adopting the same course for figures that has been recommended for their geometrical prototypes. Every figure is outlined independently of all the others, and in its natural proportions, just as if it occupied the centre of the picture. In order to see it correctly the spectator has to stand opposite to it, and as he cannot of course stand exactly opposite to more than one figure at a time, it follows that he can never see more than one at a time in correct drawing. All the others are distorted by foreshortening. But if he is at a distance from the picture this distortion is not noticeable, and when he comes near he confines his attention to the figure nearest his eye.

This is very well illustrated by Raffaelle's fresco called the "School of Athens," of which the principal part is shown in Fig. 63. The figures on the extreme right and the spheres which they carry are drawn, as has been pointed out by M. Thiébaut, just as if they were in the centre of the picture, opposite the eye. If the spectator were in the room, one wall of which is occupied by this picture, the

Fig. 63. The School of Athens.

chief one of these figures would appear foreshortened, as in Fig. 64, *a*, the sphere looking like an egg on end. One sometimes in the scenery of a theatre sees a round ball on a post present a similar aspect, from the same cause. In order to make the figure appear as it should, to make it assume when seen from the middle of the room the form intended by the painter, he would have had to draw it as shown in Fig. 64, *b*, giving the sphere a flattened form, just as in Fig. 59 *c*.

Fig. 64, *a*.

Fig. 64, *b*.

The architectural background.

267. If his picture is a large one, the painter often has a difficult task to reconcile his background and accessories, which are drawn according to perspective rule, and calculated to be seen from a single point, with his figures, which utterly violate these rules, and permit and indeed require the spectator to regard them from half a dozen different positions. The background proper may not give much trouble. More of it will be seen between the figures than would be the case in nature, as has been pointed out already in regard to columns, and care must be taken not to use any forms which require the spectator to remain exactly at the station point; for this he will not do. But that can easily be managed. The chief difficulty is found in fitting the figures into the foreground. If a chair, for instance, occupies one end of the front of the picture and is put into perspective along with the walls and floor, so as to appear correctly from a point opposite the middle of the picture, it must needs look more or less crooked when looked at from a point opposite the end

of the picture, and it is no easy matter to make a figure painted from that point of view look as if he were seated comfortably in it. If, moreover, he is to be represented as looking straight across the picture, it is by no means clear whether he should be drawn in profile, as he would appear from this point, or with the three-quarter face which he would show from the other.

268. There is even greater difficulty in reconciling the perspective of a floor with the want of perspective of the feet that stand upon it. If a number of persons are shown in the foreground of a picture all facing the same way, it is impossible to make the direction of their feet agree with that of the boards on which they stand.

Fig. 61, *a*.

Fig. 61, *a*, shows how the feet of a dramatic company, seen just as the curtain is descending, would be drawn in true perspective, agreeably to the perspective plan below. Fig. 62 shows the necessary correction, each pair of feet being drawn just as if it were exactly opposite the spectator. But it is to be noticed that the end man is necessarily represented as standing diagonally across the floor boards.

Fig. 62.

269. In general, the attitude of the figures at the edge of a large picture is not very clearly defined, and varies as the spectator changes his position. In Guido's "Aurora," for instance, if one stands opposite one end of the picture the figure at the extreme left seems to be marching along the front, and Aurora herself to be looking out of the frame. If he goes to the other end she seems to be looking back at Apollo in the chariot, and

the other figure seems to be just coming round the corner, so to speak, from behind his car. The analogous phenomenon of a portrait seeming to follow one with its eyes is commonly observed. It is not so frequently recognized that the whole face seems to turn, especially when a front face is shown, the cheek, not the nose, seeming to be the most prominent feature.

In the upper part of very high pictures the heads have to be drawn as they appear when seen from beneath, the under side of the jaw and of the eyebrows being shown. But this makes the head seem to fall backwards as one recedes from it. In Titian's "Assumption," for example, the attitude of the head varies greatly according to the position of the spectator. From the opposite end of the room in which it hangs, the face seems to be turned up and the head thrown back. As one approaches the picture it seems to bow forward.

Another difficulty in the perspective of figure-subjects is, that if the figures are as large as life, everything nearer than they are becomes colossal. But this may be avoided, and generally is, by not having anything in particular in front of the principal figures.

270. All the difficulties encountered with the human figure are met in even greater force with figures of animals, except that it is possible for them to be considerably out of drawing without detection. Fig. 65, which is borrowed from the work of M. Thiébaut, illustrates at once the extent of the distortion and the difficulty of correcting it. The form of the

Fig. 65.

pedestal controls the position of the horse's hind legs, and necessitates a distortion for which there seems no remedy. A distortion which is hardly noticed in a four-legged table is intolerable in a quadruped. They should never be drawn in parallel perspective.

CHAPTER XIII.

CYLINDRICAL, CURVILINEAR, OR PANORAMIC PERSPECTIVE.

271. THE previous chapter has discussed the so-called distortions to which circular, cylindrical, and spherical objects are subjected when drawn according to the methods of plane perspective, and has explained the so-called corrections which are applied to such objects. Similar distortions, it was shown, attend the putting of the human figure and animals into perspective, and similar corrections apply. Indeed, it was pointed out (260) that every object not exactly at the centre of the picture must necessarily be more or less out of drawing, though the distortion is not generally such as to attract notice save in the cases mentioned.

Plate XV. Distortions of rectilinear objects.

272. Plate XV. demonstrates the existence of these distortions, exhibits some instances in which they are intolerable, even in the case of rectilinear objects, and shows yet another way of correcting them. By *distortion*, as has been said, we mean that the outline given in the drawing is different from the outline presented to the eye by the object drawn. Now the rays of light that pass from the outline of an object to the eye form an irregular cone, whose base is this outline itself (255). The perspective repre-

sentation of this outline is the line in which this cone of rays is cut by the plane of the picture. If this plane cuts the cone of rays in a direction at right angles to its axis — that is to say, if the object is at the centre of the picture — then the section is of the same shape as the base; the perspective is of the same shape as the object. But if the plane of the picture cuts the cone of rays obliquely — as must be the case with all objects not just at the centre — then the section is not of the same shape as the base, and the perspective does not look like the object; it is, so to speak, distorted. Of course, when seen from the station point, obliquely, the perspective is *foreshortened*, and looks just as the object does. But in itself, and when looked at merely as a line, it presents a different form (224, 225, 256, 260).

273. This is illustrated in Fig. 70, in which is seen, at *b*, a rectangular block, drawn in parallel perspective, but considerably to the right of the centre.

Fig. 70.

Its aspect is such as no rectangular block could ever possibly present to the eye. It exhibits three faces, one of which is a square. But if a rectangular block is held so that one of its faces shows four right angles, it must be held so that neither of the other faces can be seen at all. If, on the other hand, it stands so that all three faces are seen, as this block evidently does, then all the angles must appear either acute or obtuse. The figure within the circle shows how such a block really looks when one looks straight at it, and this is the way it is drawn when at the centre of the picture. The difference between these two representations exemplifies the

distortion to which all shapes are subjected when the line from the object to the eye is not at right angles to the picture.

But this distortion in the drawing is corrected, by foreshortening, when one looks at the drawing from the station point, S, which in this case is a few inches in front of V^C, in Fig. 66. In fact, Fig. 70, *a*, was sketched from this point, and is a view, not of the cube itself, but of the Fig. 70, *b*, thus foreshortened into a real likeness of the object it represents.

Fig. 70, *a*.

Fig. 70, *b*.

274. Fig. 66 exhibits other and even more striking phenomena. Take first the church on the left hand. It is horribly out of drawing, although the picture does not extend, on this side, very far from the centre. Not only is the church and the belfry twisted out of shape, but their dimensions increase instead of diminishing as they recede from the eye. This sort of distortion is often seen in old-fashioned prints, and in photographs of very long buildings, or of interiors, taken nearly in elevation. It arises, as is obvious, from both vanishing points, V^R and V^L, being on the same side of the object.

A street.

275. It is accordingly a maxim in perspective that the objects represented must lie between their principal vanishing points, and that the plane of the picture must be so taken as to effect this. In other words, the angular range of the picture to the left of the center, V^C, must not exceed the angle between the plane of the picture and the right-hand side of the object, and *vice versa*

In the figure, for example, the front line of the church and houses makes an angle of about 20° with the picture. V^L is accordingly about 20° to the left of C, and that is as far as the picture of objects whose lines are directed to V^R and V^L, like these, can be extended in that direction.

276. If then it is desired to embrace a considerable extent of horizon, and at the same time to represent objects as being nearly parallel to the picture, there is no choice but to take the plane of the picture in such a way that they shall be *exactly* parallel. That is to say, Parallel Perspective must be employed. For where a rectangular object has one of its sides very nearly parallel to the picture the horizontal lines of the other side must be nearly perpendicular to it, and their vanishing point very near the centre, V^C. Unless, then, the centre is set quite at the edge of the picture, which is undesirable, the distortion shown in the figure must occur.

Interiors.

If, for example, the opposite sides of a room are both to be shown at once, it will not do to set the end of the room at an angle with the picture, however acute; for the end wall will contain the vanishing point of lines parallel to the sides, and one of the side walls that of lines parallel to the end. The other side will then be beyond both vanishing points, and must experience this disagreeable twisting, as is often seen in paintings and photographs.

Streets.

It is the same with street scenes. If both sides of a street are to be seen at once, they must be perpen-

dicular to the plane of the picture. Fig. 66 sufficiently shows what will happen if they are not.

277. The distortions at the other end of the picture, however, though less offensive, and consequently much more common, are almost as great. The buildings are too long and the hills not steep enough. Both have quite different proportions from what they would present to a spectator at S, and are quite unlike any sketch that would be made of them from that point. For the proportions of an object, that is to say, the relative size of its parts, depend upon the relative angular dimension of the parts, that is, upon the relative size of the angles they subtend at the eye. The apparent distance apart of the points that define them right and left, or up and down, is angular distance. A painter, then, who would represent things in their true proportions, as they look, and in their apparent relations one to another, would have to proportion the linear dimensions upon his canvas to the angular dimensions of his object. And this, in fact, is just what every painter, every draughtsman of whatever kind, always does when he undertakes to sketch from nature. It is the method of every artist who undertakes, outdoors or in, to draw things as he sees them; he can have no other; he must give to the representation of objects the apparent shape and the relative size that the objects themselves present to his eye. In other words, he proportions the linear dimension upon his canvas to the angular dimension of the object. But this is exactly what perspective does not do. In sketching, one may

Angular dimensions.

begin in the middle, fix the position of his central object, and he will naturally distribute other things about it to the right and left, according to their apparent distance from it. He proportions their distance to their angular distance, and their size to their angular dimensions; that is, to the difference of the angular distance of their edges. But in a perspective drawing, as is clearly shown in the plan, the distance of an object from the centre of the picture is not proportional to its apparent, or angular, distance, but is constantly greater, being proportionate to the *tangent* of the angle, and its size is accordingly proportioned, not to its angular dimension, but to the difference of the tangents of the angular distance of its edges from the centre. The scale to which they are drawn accordingly increases from the centre outward, just as in Mercator's Projection, which gives, indeed, a sort of perspective view of the terrestrial sphere, as seen from a station point at its centre.

278. In view of these evils, and of the practical impossibility of escaping them by using the station point, the attempt has been made to avoid them by representing objects just as they appear, *making the linear dimensions in the drawing proportional to the apparent or angular dimensions in nature.* It is plain that this scheme could be thoroughly carried out only by drawing on the inside of a hollow spherical surface, a condition impossible to fulfil. A cylindrical surface, however, answers nearly as well, especially when, as is usually the case, the vertical dimensions are relatively small. A cylinder, moreover, has the advantage of being a *devel-*

opable surface; it can be rolled out flat. This is the surface employed in circular panoramas, and it is virtually that employed in sketching from nature. For as one turns from one object to another he virtually keeps the corresponding part of his canvas directly in front of him, at right angles to his line of vision, just where a cylindrical surface would be.

279. The plate illustrates the result of this procedure, and affords an opportunity of comparing it with the results of plane perspective. In the plan of the street we have the position of the spectator indicated at S; that of a transparent plane, representing a picture plane, at p V^c p, and that of a transparent cylinder at a V^c b. The centre of the perspective picture is at V^c, the point nearest the spectator, and the plane and cylinder are tangent at that point. Visual rays drawn from the principal points in the street to the station point pierce both surfaces, and pictures drawn upon them would, when seen from the point S, obviously coincide with each other and exactly cover the objects represented.

Projection upon a cylinder, instead of upon a plane.

Figs. 66, 67.

280. Fig. 66 exhibits the result, as shown on the plane p p, and Fig. 67 that shown on the cylinder a b. The first strikingly illustrates what has been said of the inevitable distortion of objects in plane perspective, and of their gradual exaggeration of scale as they recede from the centre. Fig. 67 shows the effect of making the linear dimensions in the drawing correspond to the angular dimensions of the objects drawn, that is to say, of drawing everything just as it appears. Of these

effects the most noticeable are these: that in the first place the distortion of the church on the left entirely disappears, and in the second place the distortion on the right disappears also, the houses and the landscape beyond being reduced to dimensions proportioned to the dimensions given to the nearer objects, while the size of the picture is greatly diminished. All this is a great gain. But on the other hand the horizontal parallel lines are all more or less curved. The lines which in

Straight lines drawn as curves.

Fig. 66 are all straight and converge to a single vanishing point, in Fig. 67 converge towards these two vanishing points. Indeed, every horizontal line, except those in the plane passing through the eye, is drawn as curved. Its perspective lies in the line in which the plane passing through the eye and the line itself intersects the cylinder. This is an ellipse in space, and its development is a curved line, concave towards the horizon, which it crosses at points 180° distant from each other, the perspective of its vanishing points. This curvature would of course disappear if the paper were bent into a cylindrical form, and the eye placed at the axis of the cylinder, opposite the horizon, and in the large circular panoramas which are sometimes exhibited, and which have given to this method the name of Panoramic Perspective, this of course is done. But in general the developed cylinder has to remain flat, and it must be confessed that this curvature of lines, which in nature are straight, is itself a distortion which most persons find extremely objectionable.

281. It is worth while to remark, however, that this phenomenon of the apparent curvature of straight lines is of constant occurrence in nature; and it is just one of those phenomena of nature with which perspective has to do, being concerned with the appearances of parallel lines. All systems of lines which are long enough to indicate both their vanishing points, converging to one point on the right and to another on the left, have an apparent curvature. Such, as has already been pointed out, are the long parallel lines of cloud which often cover the sky, or the sunbeams and shadows which sometimes at sunset pass completely over from west to east. In both these cases each particular cloud or sunbeam, as one looks at it, seems quite straight; but all the others on either side seem concave towards it. In fact, as they all meet, or tend to meet, at two different points, and to separate between them, they *must* seem curved; *straight* lines can meet at only one point.

Straight lines often seem to be curved in nature.

It is the same with the horizon itself, which seems straight when one looks at it, but seems curved when one looks up or down. So with other long lines, such as eaves, sidewalks, or housetops. As one turns his eye rapidly from one end of a street to the other, the apparent curvature reveals itself unmistakably.

Now as all straight lines in nature, if prolonged, *seem* to be curved, approaching the horizon at the gradually increasing angle, so in Panoramic Perspective, their perspective representations *do* curve, until they cut the horizon.

282. To one who is accustomed to observe this curious phenomenon, the curvature of the lines in Cylindrical, or, as we may now call it, *Curvilinear*, Perspective is but a trifling evil, hardly to be counted against its manifold advantages. Of these the chief is perhaps, as has been said, the perfect conformity of its results with those obtained in sketching from nature. Of this an excellent illustration is afforded by Fig. 68, a rude outline sketch from a water-color by Turner, representing the Ducal Palace at Venice, and the adjacent buildings. He sketched each building just as it looked, and did not mind the resulting curvature of the horizontal lines of his drawing.

Fig 68.

This drawing exhibits, however, what is perhaps as objectionable a distortion as any, an apparent convexity in the objects represented. The quay, which in fact is straight, looks convex, as does also the village street in Fig. 67.

In photographs taken with a revolving camera, the pictures are virtually taken upon a cylindrical surface, and exhibit the same phenomena. Photographs taken with a common camera are in Plane Perspective.

Photographs.

283. This convexity is not, however, very noticeable, except in cases like these, where a long line is parallel to the cylinder near the middle of the picture, its perspective being horizontal in the middle and curving to the right and left. Where the subject is so chosen that horizontal lines occur only at the ends of the picture, so that the lines curve only one way, both the curvature

and the convexity are less noticeable; and they can hardly be detected where the lines are short and broken. In such cases the special characteristic of Curvilinear or Panoramic Perspective, that it permits the limits of the picture to be extended indefinitely without the rapidly increasing distortions to which Plane Perspective is liable, can be taken full advantage of. A drawing in Curvilinear Perspective may often be made to embrace a hundred, or even a hundred and twenty degrees of horizon, with less embarrassment than is in Plane Perspective incurred by sixty.

The extension of the range of the picture.

Fig. 69, which is borrowed, though much reduced, from a rare and little-known work, by Mr. W. G. Herdman, published in Liverpool in 1853, exhibits this excellence in a striking degree. It represents the meeting of two streets in some foreign town, and succeeds in showing both sides of both streets, without distorting any part of either. The horizontal angle embraced must be more than a hundred degrees. Most of the dotted curved lines, which in the original were carried across the picture in order to show the theory on which the drawing is constructed, are here omitted. Where, as here, the sky-lines are broken, the principal objects in angular, not in parallel perspective, and the continuous horizontal lines few, the disadvantages of this method, as is evident from the figure, are reduced to a minimum.

Fig. 69.

284. Whether in any given case plane or cylindrical perspective is to be preferred is a matter of judgment, and one's decision must depend chiefly upon the nature

of his subject. For architecture, except in picturesque sketches, the latter is in general obviously unfit; but for the landscape painter it affords the same means of escape from the inevitable distortions of plane perspective that the painter of figures finds in the corrections described in the previous chapter.

285. The application of Plane and of Cylindrical Perspective to the same object giving such different results, and Cylindrical Perspective being, as we have seen (277), the method one naturally adopts in sketching from nature, it follows that any attempt to apply the principles of Plane Perspective to such sketches must lead to confusion. This attempt is, however, constantly made: artists, in trying to avail themselves of the materials they have collected in their note-books, frequently resorting to the rules of perspective to correct the inconsistencies and errors, and to fill in the omissions of the originals. The vexation and trouble into which this inevitably brings them — for a drawing made upon a cylinder cannot be treated as if it were made upon a plane — has produced a very general impression among them that the principles of perspective, though true in theory, as they say, are practically false; that they doubtless work very well for geometrical work, like architectural drawings, but that when applied to nature, to the delineation of real things, even to real buildings, they break down.

Rectifying sketches.

286. It is indeed plain that such sketches must be interpreted in the light of the system according to which they were made. It is the principles and rules of Cur-

vilinear or Panoramic Perspective that must be called in. This being so, it is worth while to inquire on what geometrical principles this system rests, what are its practical methods, and what are its relations to the system of Plane Perspective.

287. If the picture is supposed to be drawn not upon a vertical plane, but upon a vertical cylinder, with the spectator at the centre, or axis, it is plain that, so far as concerns the horizontal dimensions of objects, their perspective representations will be exactly proportionate to their apparent angular dimensions, a given linear measure will correspond to a degree of arc on the horizon, and a given length of horizon in the picture to the whole circumference of 360°. Every part of this horizon will be equally near the station-point, and every point in it may, in turn, be considered as the Centre. We thus entirely avoid those distortions which, in Plane Perspective, necessarily result from the visual rays crossing the plane of the picture at an acute angle, as they must do everywhere except just opposite the station-point. All the rays, at least the rays coming from all points of the horizon, cross the picture at right angles; everything is, so to speak, at the Centre.

Geometrical principles.

And just as in Plane Perspective the perspective of a straight line is the line in which the plane of the picture is intersected by a plane of rays passing through the line and also through the eye, or station-point; and just as the horizon of every system of planes is the intersection of the plane of the picture by a plane passing

through the eye parallel to the other planes of the system; so now the perspectives of lines, and the horizons of systems of planes, are the lines in which these planes of rays intersect the *cylinder* on which the picture is to be drawn. This, which has already been illustrated in Figs. 66 and 67, Plate XV., is more fully set forth in Fig. 71, Plate XVI. Here S, in plan and elevation, represents the station-point in the axis of the cylinder of the picture. If, now, *a*, *b*, *c*, and *d*, in the elevation, are parallel horizontal lines, shown in plan at L, the plane of rays lying between them and the eye will cut the cylinder A at a', b', c', and d', the line of intersection being an ellipse, or, at c', a circle. If the cylinder is turned a quarter round, as at B, these lines of intersection will appear as semi-ellipses at b'' and d''; the semi-circle c', however, appears still as a straight line at c'', while the line a', lying at an angle of 45°, appears as a semi-circle at a''.

Plate XVI. Fig. 71.

In like manner the line *e*, at right angles to these lines, and shown in plan at R, gives the semi-circle e'' in A and the line e' in B.

288. If now the cylinder is developed, these lines of intersection will appear as in Fig. 72. The circle of the horizon at *c* will become a straight line; the vanishing-points v^l and v^r, $v^{l\prime}$ and $v^{r\prime}$, will appear at V^L and V^R, $V^{L\prime}$ and $V^{R\prime}$, 90° apart; while the semi-ellipses a'' b'' d'' e'' will be developed as the curves a''' b''' d''' e''', parallel to the horizon at their highest point, representing the point where *a b d* and *e* are nearest the eye, and converging towards

The development of the cylinder.

Fig. 72.

their vanishing-points, as parallel lines ought to do. The height of the lines a''' and e''', representing a and e, which lay 45° above the horizon, is equal to the radius of the cylinder. If the secant planes, $a'\,b'\,d'$, Fig. 71, had been carried entirely across the cylinder, forming whole ellipses, as shown by the dotted lines, these curves would be continued beyond the point $V^{L'}$ on the opposite side of the horizon, as far as the point $V^{L''}$, which is, of course, the same point as V^{L}, being 360° distant from it.

289. These curves, always concave towards the horizon, with points of contrary flexure where they cross it, and points of maximum curvature where they are at their greatest distance from it, are what are called in geometry sine-curves, or cosine-curves, and are similar to those obtained by the projection of a regular spiral. It will be observed that the angles at which they cross the horizon are the same as those in Fig. 71, A, and measure the angular distance of the lines represented above the horizon at their nearest point. The maximum distance of the perspective of each line from the horizon is proportional to the tangent of this angle.

Sine-curves.

290. The perspectives of vertical lines and the traces of vertical planes are of course vertical, and these lines are straight, being elements of the cylinder, and they remain straight when the cylinder is developed. The perspectives of inclined lines and the traces of inclined planes are sine-curves, since they too are the lines in which the cylinder is cut

Vertical and inclined lines and traces.

by planes; but their vanishing-points, instead of being upon the horizon, are above and below it, as, for example, V^N and $V^{N'}$ in Fig. 72. But they are all parallel to the picture at the point where they are nearest to it and to the eye, half way between their vanishing-points, and there have their real inclination.

Systems of sine-curves.

291. It is a property of a *system* of sine-curves, — that is to say, a series of curves crossing the horizon at the same point, like the perspectives of a system of parallel lines, — that vertical lines drawn across them are divided proportionally to the maximum heights of the curves, and that for each vertical line the tangents to the curves, at the points of intersection, meet at the same point on the horizon. It is necessary, then, to construct only one line of such a system by developing the line of intersection. The position of any other line, of which the angular distance above the horizon, and consequently the maximum height, is known, may be found at any point by drawing a vertical line, and its direction at that point may be found by drawing a tangent. In this respect a number of sine-curves drawn between the same two points are analogous to a number of ellipses with the same major axis (265), as illustrated in Fig. 48, Plate X.

To construct the sine-curves.

Fig. 73.

In Fig. 73, for example, the distances cut off by the curves upon $h\,h$ are proportional to those cut off upon $t\,t$, as may be seen at $t'\,t'$, and the tangents at the points of intersection all meet at V^L_P. These tangent lines lie in a plane tangent to the cylinder, before

it is rolled out flat, along the line $t\,t$. This relation is shown in Fig. 71, Plan, from which it is clear that these tangent lines are the perspectives upon the plane $p\,p$, regarded as a plane of the picture tangent to the cylinder at the point t, of the same lines whose perspectives are given upon the cylinder by the curves. Of course they meet at a point. This point, V^L_P, is then the vanishing-point in Plane Perspective of the same lines whose vanishing-point is found in Curvilinear Perspective at V^L_C. The plan in Fig. 66, Plate XV., exhibits similar relations.

Plane Perspective as an auxiliary method. One sine-curve sufficient.

292. These considerations make it practicable not only to draw all these curves without difficulty when one of them has once been constructed, but in practice to dispense with all but one entirely, treating each separate portion of a panoramic picture as if it were drawn on a vertical plane tangent to the cylinder in that place.

Fig. 74.

This is illustrated in Fig. 74, *a*, where we see, as in Fig. 66, an object, L R, the station point at S, and, between them, both a cylinder and a plane of projection. The object not being very large, its picture on the cylinder will not differ perceptibly from its picture upon the plane; and the latter, being easier to draw, may be substituted for the other, using V^L_P for V^L_C, and V^R_P for V^R_C.

No sine-curve necessary.

293. Fig. 74, *b*, shows that, in laying out a panorama, it is not necessary to draw even a single sine-curve, the points V^L_P and V^R_P being ascertained

directly from V^L_C and V^R_S, in accordance with the following rule: —

To find the auxiliary vanishing-points.

Given the Horizon, the vertical line of tangency, and the vanishing-points on the cylinder, set off below the Horizon, on the vertical line of tangency, the radius of the cylinder; with this radius describe an arc tangent to the Horizon; on this arc lay off, right and left, the distances of V^R_C and V^L_C; prolong the radii through the extreme points thus attained, and the points where they strike the Horizon will be the vanishing-points required, V^R_P and V^L_P, on the Horizon of the tangent plane. The rest of the construction can then take place as in Plane Perspective.

This rule explains itself if we observe that Fig. 74, *b*, is the same as Fig. 74, *a*, with the two parts brought together.

Fig. 75.

294. Fig. 75, which is taken from the etching of Greenwich Hospital, in Turner's "Liber Studiorum," further illustrates the occurrence of curved lines in sketching from nature, when every object is drawn of just the shape and relative size that it presents to the eye; that is, in proportion to its angular dimensions.

The use of a horizontal cylinder.

295. In a drawing upon a vertical cylinder, although the horizontal dimensions are proportional to the angular dimensions of the objects represented, vertical dimensions are proportional to the tangents, as upon a plane. If it were practicable

to draw upon a spherical surface, this would not be so, the linear dimensions in every direction being made proportional to the angular dimensions. The employment of a horizontal cylinder to draw upon, or in, would of course effect this for vertical dimensions. This is practicable for isolated objects, and the result may sometimes be seen in sketches of lofty buildings or towers. The curvature of the lines, and the apparent convexity of the object, making the tower look as if it were strutting or leaning over backwards, are, however, quite as offensive as in the case of horizontal lines. But here, as in the case of the street shown in Fig. 69, these defects disappear when the object is of irregular outline; and this method has the advantage, of course, of giving the actual *aspect* of a tower or spire from a given point, with its foreshortening and altered outline, better than the other. Fig. 76 illustrates this, showing at A a view of the spire of the Central Church in Boston, taken from a photograph, and consequently drawn on the principles of Plane Perspective. At B is a sketch of the same spire, taken at a corresponding distance, drawn upon the surface of a horizontal cylinder with a revolving camera obscura.

Fig. 76.

CHAPTER XIV.

DIVERGENT AND CONVERGENT LINES. — REFLECTIONS. — SHADOWS BY ARTIFICIAL LIGHT.

THE phenomena with which perspective has to do are mainly the phenomena of parallel lines, — lines which seem to meet at an infinitely distant, or vanishing-point; and we have seen that the discussion of such lines prepares the way for the elucidation of the rather complicated phenomena of shadows, since the rays of the sun, and the shadows caused when these rays are interrupted, are also parallel right lines belonging to a single system. But it is worth while to examine also the phenomena of divergent or convergent lines, — of lines that is to say, which actually do meet, or tend to meet, at a point which is within a finite distance. These phenomena so closely simulate the phenomena of parallel lines that we may well inquire just how far the resemblance extends. Moreover, just as the sun and moon cast parallel shadows, every terrestrial source of light, being at a finite distance, casts divergent shadows, and it is a matter of practical importance to find out just how they go, and what becomes of them.

296. Now just as parallel lines *seem* to converge, and look exactly as if they were radiating in every

direction, from some point not very far off, situated upon the line between the eye and the infinitely distant vanishing-point, — so lines that *do* radiate from a point more or less distant look exactly as if they were parallel lines going off to infinity. And as a perspective drawing has to do only with the appearances of things, it represents the two classes of phenomena in exactly the same way. All it can do, in either case, is to show a system of right lines, all meeting, or tending to meet, at the same point. The point which thus simulates a vanishing-point we will call the *apex* of the converging lines.

Convergent lines.

The apex.

297. This consideration leads at once to the solution of a question in itself somewhat puzzling. If the apex, or point in which the lines meet, is in front of the spectator, it is easy enough to draw the lines of the convergent system radiating from the perspective of the apex, like the spokes of a wheel. But what shall we do if the apex is behind us, the lines being divergent, passing, so to speak, over our head and around our shoulders, and, as they recede from us in front, separating from each other more and more. Such are the rays emitted by a candle set behind one's back, and the shadows cast by it. What becomes of these divergent beams? Where do they seem to go? How shall their perspectives be drawn — so much of them as extends beyond the plane of the picture — upon that plane?

The apex behind the spectator.

298. The observation made just now as to the exact resemblance between the phenomena of divergent or

convergent lines, and the phenomena of parallel lines, affords a hint of the answer to this question. Parallel lines have two vanishing-points, 180° distant one from the other, upon a line passing through the station-point. If the spectator, looking in the direction of the parallel lines of the system, sees one of these vanishing-points directly before him, the other will be directly behind him; and on turning right-about-face he will see that instead. So with the rays of the sun. One vanishing-point is in the sun itself; the other, as we have seen, is in exactly the opposite direction, namely, in the shadow of the spectator's head (170).

But as divergent lines exactly simulate these phenomena, and cannot in fact be distinguished in appearance from parallel lines, except by some extraneous indication, they too will seem to have a second point of convergence, opposite the apex, towards which they seem to tend; and when the spectator turns his back upon the apex, and looks in exactly the opposite direction, he will see the position of this false apex directly before his eyes. So with rays of a candle, or any source of artificial light. If he faces it, the rays converge upon it. If he turns his back upon it, they seem to be directed towards the point exactly opposite the candle. This point, as before, is to be found in the shadow of the spectator's head, in that part of the shadow, namely, that we may call the shadow of his eye. That lines diverging from a point behind the spectator must seem to converge towards a point in front of him is plain; for if we suppose a line

The false apex.

to pass through the eye, in any direction, all lines lying in planes that pass through this line will seem to be directed towards one or the other of its vanishing-points. But the line drawn through the station-point from the apex of these divergent lines is such a line passing through the eye, and the divergent lines are lines lying in such planes. They necessarily tend, either way, towards the vanishing-points of the line. Of these vanishing-points, one will be in the direction of the apex, the other in exactly the opposite direction. This will be the false apex.

299. Fig. 77, Plate XVII., illustrates these points, and also brings into notice the curious phenomena that present themselves when the apex, or point of convergence, is neither behind the spectator nor in front of him, but, so to speak, along-side; that is to say, is just as far from the plane of the picture as he is himself. Let S be the station-point, *p p* the plane of the picture, and 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, ten right lines lying in a horizontal plane somewhat below the level of the eye, one end of each touching the line *p p* in plan, and the ground-line, *g l*, in the perspective view above. A simple inspection of the figure shows that the parallel lines 1, 2, and 5 have their vanishing-point at V^L; 3, 4, 6, 8, and 9, which are perpendicular to the plane of the picture, have their vanishing-point at V^C; while 7 and 10 have theirs at V^R. Moreover, the lines 1, 3, and 7, which converge upon A_1, behind the station-point, and which accordingly appear to the spectator at S to be diver-

Plate XVII. Fig. 77.

The apex in three positions.

The apex behind the spectator.

gent, have their perspectives directed towards A_1', a false apex, a point in space exactly opposite the apex itself, and as far above the horizon as the apex A_1, on the ground, is below it.

The lines 5, 6, and 7, which converge upon A_2, the apex in front of the spectator, are shown in perspective meeting at the perspective of A_2.

The apex in front.

300. Finally, the lines 5, 8, and 10, which have their apex at A_3, a point just as far from the plane of the picture as the station-point, have their perspectives *parallel*. The same is true of the lines 2, 4, and 7, which converge upon the horizontal projection of the station-point, at S. But these last lines are not only parallel, but *vertical*.

The apex along-side.

301. It appears, then, that if converging lines have their apex, or point of meeting, in front of the spectator, their perspectives will converge upon the perspective of their apex; if it is behind the spectator, they will converge upon a point which we have called their false apex, situated exactly opposite the apex itself, and which is the vanishing-point of a line drawn through the apex and the station-point; if the apex is neither in front of the spectator nor behind him, but just as far from the picture as is the station-point, the perspectives of the convergent lines will be parallel. Their position depends upon the height at which the apex lies, and is most easily determined by finding the perspective of that element of the convergent system that is perpendicular to the plane of the picture (such as line 8 in the figure), and making the rest parallel to that. Finally,

if the apex is directly above or below the station-point, the perspectives of the convergent lines will be vertical. If it is on a level with his eye, they will be horizontal.

302. Fig. 78, a sketch, somewhat reduced in scale, from an old English print, illustrates this last point. It shows three streets converging upon the point occupied by the spectator. The axes of these streets are all drawn vertical and parallel.

Fig. 78.

A similar result would follow if one were to draw the spokes of a large wheel while standing upon the hub. They would all be vertical and parallel. So with the reflections of distant lamps which at night often seem to cross the water and converge below the feet of a spectator standing upon a bridge. In perspective they also would all be drawn vertical and parallel, continuing, indeed, the vertical line of the lamp-posts from which they emanate, as is seen in Fig. 79.

Reflections.

Fig. 79.

303. Fig. 80 shows that the position of the false apex may most conveniently be determined by finding the perspectives of two horizontal elements of the converging series, *a* and *b*, one perpendicular to the plane of the picture, and one passing beneath (or above) the station-point. Let *a′* and *b′* (or *a″* and *b″*) represent the points in which these lines pierce the plane of the picture, their distance from the horizon depending upon the position of the apex, A, below or above the eye, and the other element of their position being obtained from the orthographic plan.

To find the false apex by perspective.

Fig. 80.

The perspective of the line *a* will begin at *a′* (or *a″*), and be directed towards its vanishing-point, V^C; that of the line *b* will begin at *b′* (or *b″*), and be vertical, since it passes directly below (or above) the station-point (301). The point A′ (or A″), where these perspectives would meet, if prolonged, is the position of the false apex.

The same result may be obtained by the methods of Descriptive Geometry, as may be seen in the same figure. The false apex is the vanishing-point, as has just been seen (298), of that element of the converging system that passes through the eye. But the vanishing-point of a line passing through the eye, and which is consequently seen endwise, is the point where it pierces the plane of the picture. Now *b*, in Fig. 80, is the horizontal projection of such a line, and *a′* A′ (or *a″* A″) its vertical projection. A′ then (or A″) is the point where it pierces the picture, its vanishing-point, and the false apex in question.

Another way, by descriptive geometry.

304. Fig. 77 furnishes other illustrations of the false apex besides those already mentioned. The points of convergence, A_5, A_6, and A_7, situated behind the spectator, have each a false apex, towards which the perspectives of the converging lines are directed, as may be seen at A_5', A_6', and A_7'.

Fig. 77.

A_5, A_6, and A_7 are supposed to lie in the same plane as do A_1, A_2, A_3, and A_4, previously discussed. It will be observed that the most distant of them, A_6 and A_7, have the apex nearest the horizon, the nearer ones, A_1 and A_5, farther off, and those on a line with the spec-

tator, A_3 and A_4, at an infinite distance, the perspectives of the converging lines being parallel.

If either apex were in a different horizontal plane, the vertical position of the false apex would change accordingly, that of A^7, for instance, appearing at A_7'', if the point of convergence were as far above the spectator's level as it has been supposed to be below it.

305. In determining the form of shadows cast by artificial light we follow the same line of argument as when, in Chapter IX., we discussed the shadows cast by the sun. Here, as there, the (invisible) shadow of a point is a line drawn through the point from the source of light, and its (visible) shadow on any surface is the point where this line pierces the surface. Here, as there, the (invisible) shadow of a line is a plane in space, and its (visible) shadow upon any plane that receives it is the line of intersection of these two planes, its vanishing point being at the intersection of their traces. Here, as there, the horizon of the plane of shadow is found by drawing a line through the vanishing points of two of its elements, one being the vanishing point of the line that casts the shadow, and the other the vanishing point of the (invisible) shadow of some point in that line.

Shadows by artificial light.

306. The only difference in the two cases is that in the case of sunlight the vanishing point of this last line is always known beforehand. The shadows of all the points are parallel lines, and have a common vanishing point, either in the sun, or in the shadow of the specta-

tor's head, opposite the sun. But with artificial light the lines drawn from the source of light to the different points of the line whose shadow is to be found are not parallel but divergent. Each has a direction of its own and a vanishing point of its own. Still, all these divergent lines lie in the same plane, the plane of shadow in question, like the sticks of a fan, and any one of them will suffice, with the line itself, to determine the horizon of this plane. For this it is necessary only to select some one of these divergent lines and determine its vanishing point.

To find the vanishing point of a ray of artificial light.

Fig. 81, *a*.

307. This does not of course lie in the apex of these converging rays, the source of light itself. It must be determined just as the vanishing point of any finite line is determined, when the line is given by two points at its extremities. In Fig. 81, *a*, for instance, let M be the line whose shadow is to be cast, determined in position by the vertical lines at its extremities, which show its relation to the horizontal plane. Let A be the apex of rays, or source of light, and let a single ray, S′, be drawn through this point and some point upon the given line, say its lowest point. These two lines suffice to determine the plane of shadow, and the horizon of that plane, H S M, will pass through their vanishing points, $V^{S'}$ and V^{M}.

Fig. 81, *b*.

308. Fig. 81, *b*, shows how these may be determined. The horizontal line R, connecting the perpendiculars let fall from the ends of the line M, being prolonged until it touches the horizon, finds its

vanishing-point at V^R (for a line lying in a plane always has its vanishing-point in the horizon of that plane); the horizon of the vertical plane that contains M and R passes vertically through V^R (for the horizon of a plane passes through the vanishing-points of every line that lies in it); and the line M prolonged finds its vanishing-point, V^M, on this horizon (the horizon of the vertical plane in which it lies).

To find the horizon of the plane of shadows.

In the same manner $V^{S'}$ is found by drawing, as far as the horizon, a line beneath the ray of light, and by producing that ray, until it meets the horizon of the vertical plane in which it lies. H S M, the horizon of the shadow of M, is then found, as usual, by drawing a line between $V^{S'}$ and V^M.

The visible shadow of the line M upon the plane, S M on R L, lies at the intersection of the plane of invisible shadow with the ground plane. Its vanishing-point, $V^{SM,\ RL}$, is the intersection of their horizons; that is to say, of the horizons H S M and H R L. The visible shadow will begin where the ray S′, passing through the end of the line M, reaches the ground, and will be directed towards, or away from, this vanishing-point, as in the figure.

A similar result would have been reached, as appears in the figure, by taking rays of light that pass through other points of the line M, such as S″ or S‴. These have different directions, but lie in the same plane; this gives us new vanishing-points, $V^{S''}$ and $V^{S'''}$, but the same horizon, H S M, just as before.

A second way; the ray parallel to the picture.

309. Any other ray would answer as well, of course, as these; the choice of one or another is purely a matter of convenience. Now just as we found, in studying shadows cast by the sun, that they were most easily determined when the rays of light were parallel with the picture (the sun being neither behind the spectator nor in front of him, but on one side), so here we shall find that there is a practical advantage in selecting for use neither of the rays just now employed, but, instead, the ray that is parallel to the picture, the ray that passes through the line at a point which is just as far from the picture plane as is the apex, A, or source of light.

Fig. 81, *c*.

Fig. 81, *c*, shows how this is done. A line parallel to the horizon is drawn through the foot of the vertical let fall from the apex A until it intersects the line R, at which point another vertical is erected to intercept the line M. The line S, from A to this point, is obviously a ray of light parallel to the picture, and the line drawn through V^M, parallel to it, is the horizon of the plane of shadows, H S M.

This is exactly the method employed in Fig. 36, Plate VIII., founded on the proposition that the element of a plane parallel to the picture has its perspective parallel to the horizon of the plane (38).

Fig. 81, *d*.

310. The relations between the treatment of diverging light and that of parallel light may be illustrated by these figures. If the source of light, Fig. 81 *d*, be supposed to retreat to an infinite

distance, the vertical line dropped upon the horizontal plane will shorten until it rests upon the horizon. $V^{S'}$, $V^{S''}$, and $V^{S'''}$, will coalesce with each other and with A, which will now become V^S, and the figures will both assume the aspect of Fig. 81, *d*.

311. In casting shadows by artificial light, however it is not so important to obtain the vanishing-point of the line of visible shadow as it is in sunlight. For, in sunlight, the vanishing-point of the shadow of a line, when once obtained, serves for the shadows of all the other lines of the system, since the shadows are parallel as well as the lines themselves, and all have the same vanishing-point. But in artificial light the shadows of parallel lines are not parallel, and the vanishing-point of one is of no service in drawing the next one.

It is accordingly just as well, and it is much easier, to find the shadow cast upon a plane by a line exposed to artificial light by finding the shadows of its extreme points and drawing a staight line between them. Fig. 81 *e* shows how this may be done. The point in which the ray S′ touches the ground is one point of the line of shadow, the ray S″ gives a second point, and the point where the line itself pierces the ground gives a third. Either two of these suffice to fix the line of shadow in position and direction.

Fig. 81, *e*.

Fig. 81 *f* shows that in the sunlight, also, we can in like manner determine a shadow by points, without using the vanishing-point.

Shadows by Sunlight. Fig. 81, *f*.

This is, whether in sunlight or artificial light, the most convenient way of determining the shadow of a

curved or irregular line. It amounts, virtually, as the figure shows, to letting fall a perpendicular from each point upon the plane in question, and drawing its shadow, and then passing a line of shadow through the extreme points thus obtained, as is done in Fig. 81 *g*.

Fig. 81, g.

312. But a plane given by a line and a point can be determined by a line drawn through the point parallel to the given line, as well as by one crossing it. The same result as in the previous figures may accordingly be obtained by employing the method shown in Fig. 81, *h*.

Fig. 81, *h*. A third way; the ray parallel to the given line.

The point V^M having been determined, a ray, S^M, parallel to M, is drawn through the apex A, touching the ground at its point of intersection with the horizontal line beneath it, the line M being also prolonged to a similar point. This ray, the line M, and the horizontal line, L, joining their extreme lower points, are obviously all in the plane of shadow.

The line prolonged in the other direction, beyond M, being the line in which the shadow of M intersects the horizontal plane, is obviously the visible shadow of M on that plane, the exact length of which can be cut off by drawing divergent rays from A through the extremities of M, as in the figure.

The shadow of any line cast on any plane surface by an artificial source of light may accordingly be found by drawing a single ray parallel to the line in question, and finding the points in which both the line in question

and the ray will pierce the plane on which the shadow falls. A line joining these points will be the line in which the plane of rays passing through the line intersects this plane, and, if produced beyond the point where the line pierces the plane, will give the line of its shadow.

A general method.

This method seems less direct than that of Fig. 81 *e*, which is indeed often preferable for single lines. But the method just explained is to be preferred where several parallel lines are subjected to artificial light. For the point *x*, Fig. 81 *h*, where the parallel ray pierces the plane upon which the shadow falls, holds the same relation to one line of the system as to another, the shadows of all of them passing through it. The shadow of the line M_2, parallel to M, in the figure, is directed towards this point. It is then an apex of converging lines, and is as important and serviceable a point when the shadows of parallel lines are cast by artificial light, as their vanishing-point is when they are cast by sunlight, and answers the same purpose in saving labor.

313. *It appears then, that, in artificial light, the shadows upon a given plane of a given system of parallel lines all converge to the point in the plane at which it is pierced by an element of the system passed through the source of light.*

314. Fig. 81 *i*, shows that the method of Fig. 81 *h*, may also be used for finding shadows by sunlight, the shadow of a line in a plane being determined by means of a second line parallel with it, instead of by a ray of light crossing it.

Fig. 81 *i*. This method applied to sunlight.

We cannot, to be sure, draw this second line through the source of light, that being infinitely distant. But it is not essential, even with artificial light, that the parallel line be drawn through the apex of the divergent rays. It may be drawn through any point of any ray. In Fig. 81, *j*, for instance, a point being taken on a ray that passes from A to the upper end of M, its height above the plane is easily determined, and a line parallel to M can as serviceably be taken through that point as through A itself.

But this process can be exactly as well employed with the parallel rays of the sun, as appears in Fig. 81, *i*, where the shadow of M on the ground is again found without the aid of the horizon H S M.

Plate XVIII. The shadow of vertical lines. Fig. 82.

315. Fig. 82, Plate XVIII., illustrates the ease with which can be found the shadows of vertical lines upon a horizontal plane. The shadow of each such line is obviously a vertical plane, of which the vertical line dropped from the source of light, A, is also an element. The line joining the lower ends of these vertical lines is accordingly the intersection of the plane of shadow with the ground, and its prolongation gives the line of shadow. This has indeed been done incidentally, as was said, in Figs. 81, *e* and *g*.

The figure shows, also, that a similar procedure may be followed in the case of a horizontal line, normal to a vertical plane, the projection of the apex of rays upon that plane being known. (313.)

316. The principle of Fig. 81 *h* is illustrated in Fig. 83, where the shadows of several parallel lines are thrown upon a broken surface composed of several planes. The shadow of each line is directed, in each plane, towards, or away from, the point where a ray drawn parallel to it through the apex of rays, or source of light, pierces that plane. As the same ray is parallel, of course, to all the lines of the system, one ray suffices for them all, and their shadows, in each plane, are directed towards the same point.

Shadows of parallel lines.
Fig. 83.

The vertical and inclined lines that cast the shadow lie in a vertical plane, and the vertical and inclined rays taken parallel to them lie in a vertical plane parallel to it. The points 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, in which these rays pierce the planes on which the shadows fall, lie in the line in which the plane of rays cuts those planes.

Of these points, 1, 2, and 3 are those in which the vertical ray pierces the ground and the two inclined planes at the bottom of the wall, and the shadows of the vertical bars cast upon these planes are directed to these points; 4, 5, and 6 are the points at which the horizontal ray pierces the three vertical planes of the wall and base, and the shadows of the horizontal bars cast upon these planes are directed towards these points; 7, 8, and 9 are the points at which the inclined ray pierces these three vertical planes, and 10 and 11 those at which it pierces the two inclined planes, and the shadows of the inclined bars falling upon these five surfaces are directed necessarily towards these five points or apexes.

The shadow upon the vertical wall of the irregular line of the arch and the moulding that supports it, is found by the method shown in Fig. 81, *g*. (311.) Horizontal lines are drawn from successive points perpendicular to the surface of the wall, and a line of shadow is drawn through the shadow of the successive points.

317. It is to be noticed that if one were to stand under the arch, with his eye at the point occupied by the lamp, he would see just so much of the wall as is now in light. These processes enable us then to determine how much of what is shown in a picture is visible to the personages represented in it. In Fig. 83, for example, if we suppose the eyes of the young man standing outside the arch to be an apex of visual rays, and then find the shadow, so to speak, cast by them upon the wall, the point 12 being the projection of this apex upon the plane of the wall, we find that the light of his eyes just reaches the young woman sitting at the window within.

The most frequent examples of shadows cast by artificial light are presented in pictures of interiors lighted by candles or gaslight. It is customary to restrict the discussion of these phenomena to cases in which the source of light is, as in the figures already given, in front of the spectator. But the theory of false apexes explained in this chapter (301) enables us to illustrate also the case, equally common in actual experience, in which the source of light is behind the spectator. Figs.

85 and 86 illustrate these two cases. The shadows are cast by the method of Fig. 81, *f* and *g*.

318. Figs. 84 and 85, Plate XIX., represent the plan of a long room, and the perspective of one end of it. The line *pp*, in Fig. 84, shows the position of the plane of the picture, S is the horizontal prospective of the station-point, and A and B the projection of two gas-burners suspended from the ceiling. The burner, A, appears in its proper position in Fig. 85; its projection upon the floor is shown at A^H, upon the ceiling at A^Z, on the right and left hand walls, at A^R and A^L, and upon the rear wall, which is parallel with the plane of the picture, at A^P. These points are the points in each surface towards, or away from, which the shadows of lines normal to these surfaces are directed when this burner is lighted (314). Each is an apex of the diverging visible shadows, while A is the apex of the diverging lines of invisible shadow. The extreme point of the visible shadow of the normal line is in each case determined by the line of the invisible shadow of the extreme point of that line.

Plate XIX. Figs. 84 and 85.

In Fig. 86 the other burner, B, is supposed to be lighted; and we have the case already illustrated in Figs. 77 and 80, in which the apex of diverging lines is behind the spectator. Following the method of Fig. 80, we find that lines drawn normal to the plane of the picture from these several apexes pierce the picture at the points B_N, B^H_N, B^Z_N, B^R_N, and B^L_N, and the false apexes B', $B^{H\prime}$, $H^{Z\prime}$, $B^{R\prime}$, and $B^{L\prime}$ are then easily found.

Fig. 86.

B′ is determined by the aid of the plan, Fig. 84, the line $B_N C$ being the perspective of the ray normal to the picture, and the vertical line through *b* the perspective of the horizontal ray passing above the spectator. Their point of intersection, B′, is the false apex (303). It lies in the shadow of the spectator's head, just at the point occupied by his right eye, the eye with which he is supposed to be looking. This point is, then, in artificial light as well as in sunlight, the focus of rays of light proceeding from behind the spectator. Lines drawn in like manner from the points B_N^H, B_N^Z, etc., through the centre, C, will contain the other false apexes, and as the true apexes are either on a level with B or are directly above or below it, so the false apexes will be either on a level with B′ or directly below or above it; that is to say, they will lie in the intersection of the lines drawn through C, with horizontal or vertical lines drawn through B′.

The false apexes.

319. It will be noticed that the apex, B^P, towards which the shadows of the lines normal to the rear wall are directed, occupies the same position as the point A^P, towards which the shadows of the same lines were directed in Fig. 85. The only difference in the two cases is that the shadows in Fig. 86 are shorter than those in Fig. 85, being cut off by lines converging to B′ instead of by lines converging to A; but they coincide with them as far as they go. A^P and A being near together, the lines directed towards them intersect a good way off, giving longer shadows than do the lines directed towards B^P and B′.

Indeed, there is much less of shadow altogether in Fig. 86 than in Fig. 85, as was to be expected, the spectator's eye being so nearly in a line with the source of light. If it could coincide with it, as was supposed to happen in the case of the serenader in Fig. 83, he would see no shadows at all.

The shadow of the lantern, A, thrown upon the rear wall at A^P, or B^P, is really, of course, a little larger than the lantern, being at a greater distance from the apex, B. But its perspective is smaller than the perspective of the lantern, its distance from the station-point being relatively greater still. If the station-point were exactly on a line between the point B and the lantern, the outline of the shadow would obviously fall within that of the lantern itself.

Shadows of convergent lines by artificial light.

320. The shadows of convergent lines, whether cast by natural or by artificial light, converge, of course, to the shadow of their apex. But if the apex is just as far from the plane on which the shadows fall as is the source of light, its shadows will be at an infinite distance, and the shadows of the converging lines will be parallel to one another and to the shadow of that element of the convergent system of lines which is normal to the plane, and will have the same vanishing-point in the trace of that plane; if it is farther off, they will diverge, being directed towards a point in the plane beyond the source of light. This point is, so to speak, the *negative* shadow of the apex, being still the point where the line through the apex of the lines and the apex of the rays pierces the plane. It is the

same point that would be obtained if the two apexes should change places.

321. Fig. 87 illustrates these points, showing the shadows of several geometrical figures cast

Fig. 87.

upon the floor. The point A^H is the horizontal projection of the gas-light, A. The shadows of the lines which meet at the apex B, meet at its shadow, B'. The shadows of the vertical axes are directed upon A^H. The apexes C and C, are just on a level with A; their shadows upon the table are accordingly at an infinite distance, and the shadows of the lines that meet at these apexes are accordingly parallel in space, and have their vanishing-points upon the Horizon. Each point D is further from the table than A, and the shadows of the edges of the pyramid of which it is the apex converge towards the corresponding point D', the "negative shadow" of D, being the point upon the horizontal plane where the shadow of A would fall if D were a source of light.

In sunlight also the shadows of converging lines are parallel when the source of light is just as far

Fig. 88.

as the apex from the plane on which the shadow is cast. Fig. 88 shows how the Pyramids at sunrise throw parallel lines of shadow towards the western horizon.

The analogy of these phenomena of shadow to those of the perspective representation of converging lines (301, Fig. 77) recalls the analogy, already pointed out, between the shadows of all objects when cast upon a plane by artificial light, and their perspective representation (265, Fig. 60).

CHAPTER XV.

OTHER SYSTEMS AND METHODS.

IN the processes hitherto described every line has been regarded as a portion of an infinitely long line tending towards its vanishing-point, and every surface as a portion of an infinite plane extending to its trace, or horizon; and it is by determining the position of these vanishing-points and horizon that the position of the perspective representations of these lines and surfaces has been fixed. This way of looking at the subject involves a comprehensive survey of the phenomena in question, and leads to a proper understanding of their relations. The processes deduced from this study are also generally convenient in practice; for, though some of the vanishing-points are generally somewhat remote, still the space required for drawings executed upon the small scale commonly employed is not greater than can usually be afforded.

Before dismissing the subject, however, it is proper to consider some other methods of obtaining the same results, based upon the consideration of these same phenomena, and involving a more extended application of some of the principles already considered, — methods which under certain conditions offer considerable advantages.

Plate XX. Vanishing points disused; space economized.

322. Several of these special methods are illustrated in Plate XX. In all of them the consideration of vanishing-points and horizons is more or less dispensed with, the lines to be represented being considered merely as finite lines lying between two points, the immediate object of the processes employed being to fix the perspective of these points. In some of these methods the abandonment of the outlying vanishing-points leads to so great a reduction of the space required for making the drawing that the work is performed almost entirely within the limits of the picture itself. In executing large works, such as scene paintings or mural decorations, this is, obviously, of great convenience.

The Method of Direct Projection.

The method of direct conical projection.

323. In this method no use at all is made of vanishing-points, and no reference is had to any of the phenomena of parallel lines that are represented by means of them.

The object to be represented is carefully drawn, both in plan and in side elevation, and the plane of the picture, seen *edgewise* or in section, and the station-point are shown. By drawing lines, representing the visual rays, from every point in the object to the station-point, first on the plan and then in the elevation, and noting their intersection with the plane of the picture, the horizontal and vertical position of the perspective of every point may be ascertained, and a representation of the object obtained by drawing lines connecting the points.

Fig. 89 illustrates this method, giving at *a*, *b*, and *c*, the plan of a cross, set obliquely, and two elevations, both of which are necessary, as neither one of them exhibits all the points visible from the station-point, S, in front. Lines representing the visual rays are drawn, both in plan and in both elevations, from all the visible points to the point S, and the points where they pierce the plane of the picture, *p p*, indicated. These points, being transferred to the side and bottom lines of the figure, 89, *d*, suffice to determine the position of each point in perspective.

Fig. 89.

This kind of projection, in which the lines of projection converge to a point, instead of being parallel as in plans and elevations, is called Conical Projection, as distinguished from Orthographic.

The Mixed, or Common Method.

324. The method of Direct Projection is seldom used to determine vertical dimensions, — that is to say, to fix the position of horizontal lines, — the labor of constructing two oblique elevations being intolerable; but it is very generally employed for the determination of horizontal dimensions, — that is, to fix the position of vertical lines, the length of vertical lines being determined by means of lines of vertical measures and vanishing-points on the Horizon.

The common method.

Fig. 90 illustrates the application of this mixed method to the subject of the previous figure. The vertical lines are drawn as in Fig. 89, *d*, their position being taken from the geometrical plan

Fig. 90.

at *a*, by direct projection. Their length is determined by setting off the real heights, as given by the elevation along-side, on a line of vertical measures, *v v*, taken where the plane of the front of the cross intersects the plane of the picture. This is fixed by the point *m*, in Fig. 89, *a*. Fig. 89 also serves to determine the vanishing-points V^R and V^L, and the corresponding points of distance, D^R and D^L.

The advantages of the orthographic plan.

325. Though this method is deficient in scientific unity, an entirely different principle being employed for horizontal dimensions from that used to determine vertical dimensions, it is often very convenient in practice, especially when, as is frequently the case with buildings, a carefully drawn ground-plan, prepared for other purposes, can be taken advantage of. This is still the process most commonly employed by architectural draughtsmen for the determination at least of the main lines of their work. Points of distance, points of measures, and the vanishing-points of inclined lines, are employed, if employed at all, only as auxiliaries and alternative devices.

The advantages of the perspective plan.

326. But the employment of the Perspective Plan to determine horizontal distances, and thus fix the position of the vertical lines of a perspective drawing, as has been done in the previous chapters, is altogether preferred by the best and most recent writers, and by the best informed draughtsmen. It has the signal advantage of avoiding the confusion and error that necessarily attend the

multiplication of points of intersection distributed along a single line. Even in the figure just given, simple as it is, we find in 89, *a*, a dozen points crowded together upon the line *pp*. It is not easy, in transferring them to Fig. 90, to keep clearly in mind which is which, — which indicates a point at the bottom, which a point on the top, which belong to the front plane and which to the back. In the perspective plan, on the contrary, Fig. 91, every point is significant; there is no confusion, and, the relations of all the parts being clearly exhibited, there is much less danger of trivial inaccuracies than in a blind and merely mechanical procedure. Moreover, if the perspective plan itself becomes too crowded with details, it is practicable to make a second or a third, as has already been done in Plate III. In the case of high buildings it is usual to make a separate perspective plan for each story, those of the upper stories being drawn above them, as those of the lower stories are drawn below. These plans are always perfectly intelligible and serviceable after any lapse of time, and, as has been said, may conveniently be made on separate strips of paper, thus saving the drawing itself from disfigurement, and, indeed, protecting it from injury. These strips of paper with the plans upon them can then be preserved, and in case a second drawing for any reason has to be made, half the labor of making it will have been saved.

Fig. 91.

327. Other and incidental advantages of this method are the great facilities it offers for designing in perspective, for working up a perspective drawing from rough

sketches, and altering and adding to it at will, studying the effect of such changes as may be suggested by taste or convenience. It is also to be observed that the perspective plan takes up less room than the orthographic plan, with its system of visual rays directed towards the station-point, and this is sometimes a consideration of some importance.

Sinking the perspective plan.

328. The reason why the perspective plan is so little used, although the theory of points of distance on which it is based is perfectly familiar, is that unless this plan is sunk considerably below the picture the desired points are not very accurately ascertained, the lines whose intersections determine them cutting each other at an acute angle. Sinking the plan, however, as is done in Fig. 90 and elsewhere, entirely obviates this, and has the advantage not only of enabling one to draw it on a separate paper and preserve it for future use, as has just been suggested, but of keeping the picture itself free from construction lines.

The Method of Co-ordinates.

The method of three rectangular co-ordinates.

329. The method of Co-ordinates applies the principles of Parallel Perspective, as set forth in Chapter VII., to the solution of every class of problems. Lines parallel and perpendicular to the picture are treated as is usual in that system. Lines inclined to the picture are determined, as in the method of Direct Projection, by ascertaining the perspective of the points between which they lie, their vanishing-points

being neglected. The position of a point in space being known, the three dimensions that determine its position can easily be put into perspective, two of them being taken parallel to the picture, and the third perpendicular to it; and, the perspective of every point being thus ascertained, the lines lying between them are easily drawn.

In speaking of these three directions, at right angles to each other, it is convenient, just as we call the vertical dimension Height, to speak of the horizontal dimension parallel to the picture as Width, or Breadth, and of the other horizontal dimension, perpendicular to the picture and parallel to the Axis, as Depth.

Height, width, and depth.

330. Fig. 92 exhibits the application of this method to the same subject as that by which the other methods just mentioned were illustrated. The eye being supposed to be about two inches from the paper, the point of distance would be two inches from V^c, the centre of the picture. The point of half-distance is accordingly set one inch off, at $D\frac{1}{2}$, and the *perpendicular* dimensions are laid off upon the ground-line of the perspective plan in Fig. 92, *b*, at half the scale of the orthographic plan above (Fig. 92, *a*), from which they are taken. (142.)

Fig. 92. *a* and *b*.

In Fig. 92, *c*, the vertical dimensions, as given by the elevation in Fig. 90, are laid off upon the scale of heights erected at *g*. Horizontal lines drawn from the points thus ascertained to the centre, V^c, and vertical lines drawn from the points pre-

Vertical dimensions. Fig. 92, *c*.

viously ascertained upon the scale of depths, drawn from g to V^c, in the plan below, determine by their intersection the height above the ground-plane and the distance

The side elevation in perspective.

behind the plane of the picture of every point in the object to be represented. This enables one, if he pleases to do so, to construct a perspective of the side elevation, as is done in the figure, just as the perspective of the plan is constructed. In fact, Fig. 92, *c*, is the perspective of Fig. 89, *b*, just as Fig. 92, *a*, is the perspective of Fig. 89, *a*. The perspective plan and elevation being both given, the perspective of the object is easily constructed.

Fig. 92, *d*.

It is sometimes convenient to construct this perspective elevation in a vertical plane not perpendicular to the picture; a plane, that is, whose horizontal elements are directed to some other point of the horizon than the centre, V^c. This is shown in Fig. 92, *d*. In this case points upon a new line of depths are taken across from the line $g\,V^c$.

331. All this, though simple in theory, is laborious in practice, as the application of general methods to special problems is apt to be. In most cases it is not worth while to give up the facility and accuracy afforded by the use of vanishing-points for this tiresome and roundabout process; but when the object to be drawn is irregular in shape, or bounded by curved lines, so that it has to be put in by points at any rate, the method of rectangular co-ordinates, according to parallel perspective, best meets the case. Even when such objects occur in a drawing made in angular perspective

it is often convenient to employ it. When, finally, the scale of the drawing is so large, or, what comes to the same thing, the space to work in is so small, that the vanishing-points are inaccessible, this method is of great service. By employing points of half distance, or quarter distance, etc., the necessary constructions can generally be confined within the limits of the picture itself.

332. The most common application of the principle of co-ordinates is to the determination of the size of miscellaneous objects, such as trees, animals, or human figures in landscapes. A vertical scale being established in the plane of the picture, resting upon the ground-line, lines converging to any convenient point on the Horizon suffice to show how large any object, a human figure, for instance, should be drawn in any part of the picture.

The scales of height, width, and depth.

This use of a scale of heights is illustrated in Fig. 93. The figures are supposed to be all of the same height as the one in the immediate foreground. The scale of heights, on the left, shows how tall such a figure will appear at every point of the horizontal plane between the ground-line and the Horizon. The position of such a figure above or below that plane will not of course affect its apparent size. The man in the balcony, on the right, for instance, is drawn just as tall as the man on the platform beneath, and the persons upon the inclined plane descending to the water are of the same height as those upon the pavement alongside.

Fig. 93. Figures, etc., in a landscape.

The size of the different vessels is determined in a similar way.

333. It is worth while here to point out that though points of half-distance, quarter-distance, etc., in Parallel Perspective, do not serve, as do points of distance, as vanishing-points of lines of 45°, such lines can nevertheless easily be drawn through any point by their aid.

To draw a line at 45°, using points of half-distance, quarter-distance, etc.

Fig. 94.

Let *a* and *b* in Fig. 94 be two points through which it is desired to draw lines making 45° with the axis and with the ground-line, the centre, V^C, and the point of half-distance, $D\frac{1}{2}$, being given. By drawing through these points lines directed towards V^C and $D\frac{1}{2}$, crossing them with a line parallel to the horizon, and then doubling upon this line the distance intercepted, lines may be drawn which are obviously directed towards $D{=}V^X$.

If the point of one third-distance is given, the intercepted portion must be trebled, as at *c*, or quadrupled, as at *d*, if the point of quarter-distance is used.

It is hardly necessary to explain how a square can be erected on a given line parallel to the ground-line, as is shown in Fig. 95, using points of half, third, and quarter distance.

To draw a square.

Fig. 95.

The Method of Squares.

334. The processes of the method of co-ordinates may be much simplified, especially in the case of objects irregular in plan, by adopting the de-

Squaring.

vice of *squaring*, commonly used by draughtsmen to assist them in copying the outlines of drawings, especially such as are to be copied on an enlarged or reduced scale. It consists in first covering the drawing to be copied with a network of lines, then reproducing this network at the scale required, and finally in filling in, by the eye, the portion of the drawing included in each of the reticulations.

335. The Method of Squares applies a similar procedure to the putting into perspective of a complicated perspective plan. A network of lines being first drawn over the plan in question, its perspective representation is easily drawn in parallel perspective. The details of the plan can then be filled in with sufficient accuracy, and the vertical dimensions obtained from a scale of heights.

Fig. 96 illustrates this procedure, *a* being the orthographic plan, *squared*, *b* the perspective plan, and *c* the drawing.

Fig. 96.

The figure does not show how the heights are obtained. They may be obtained either by *squaring* a side elevation and putting it in perspective, after the manner of Fig. 92, *c*, or by erecting lines of vertical measures at convenient points in the plane of the picture, as in Fig. 90.

336. If a sunk perspective plan is used, as in the drawing, the outlines of the plan in the picture can most easily be found by the use of proportional dividers, the distances of the corresponding points from the horizon being proportional.

Mr. Adhémar's Method.

337. The system of Mr. Adhémar is an ingenious variation of the method of Co-ordinates. Like that method, it does not rely upon the use of any vanishing-points except the one at the centre of the picture, and it enables the work, if necessary, to be entirely confined within the limits of the picture itself. Such vanishing-points as lie within these limits, however, whether they belong to any of the lines that occur in the objects represented, or are merely auxiliary, like points of proportional measures or points of half-distance or quarter-distance, are made the most of.

Mr. Adhémar's system.

This system, like that described in the previous paragraphs, is especially adapted to cases in which, from the scale of the drawings, from the limitation of the space at command, or from the great distance of the station-point from the picture, the vanishing-points of the principal lines are inaccessible. It leaves one free to take the point of view most conducive to the desired result, without considering whether the making of the drawing will be more or less difficult.

338. In the application of this method Mr. Adhémar employs four special devices. These are Small Scale Data, Vertical Margins, Auxiliary Directrices, and the Inclined Perspective Plan.

339. It is obvious that the larger a perspective drawing is to be made, the more convenient will it be to draw out the data from which it

Small scale data.

is to be constructed at a reduced dimension, a half, a third, or a fourth, of that employed in the picture. Instead, however, of magnifying the data furnished by such orthographic drawings in order to bring them up to the scale of the plane of measures, as is done with Fig. 84, an auxiliary plane of measures is employed, two, three, or four, times as far off, a plane so distant that the small scale dimensions given in the data can be employed without change to establish scales of height and breadth. The measures of depth, perpendicular to the plane of the picture, are laid off upon the ground-line, as usual; but here, too, the same small scale is employed, the dimensions being transferred to the scale of depth by means of fractional distance-points (142).

Reduced scales of height and width in an auxiliary plane of measures.

Reduced scale of depth and fractional points of distance.

340. Fig. 97, *a*, *b*, and *c*, illustrates this procedure. At *a* we have first the elevation of the object to be drawn, in this case a pyramid; the line of the horizon is drawn to show how much of the pyramid is above, how much below, the eye. Alongside is an Orthographic Plan, on the same scale, showing the position of the pyramid relatively to the plane of the picture and to a line, G O, drawn upon the horizontal plane perpendicular to the picture, at a convenient distance from the pyramid, to serve as a scale of depth. The dotted lines show the distance from these two lines to the four angles of the pyramid which are visible from the station-point. They are the horizontal co-ordinates of these points. The vertical co-ordinates of these points are

Plate XXI. Fig. 97.

shown by a dotted line at the side of the elevation. V^C is, as usual, the centre of the picture opposite the eye, at S, and $D\frac{1}{2}$ is the point of half-distance.

341. At *b* we have this Orthographic Plan put into perspective, and the pyramid in perspective above it. But the scale is doubled, the distance *g c*, upon the ground-line being double that of G C in the plan. Instead of using the ground-line, however, as a line of horizontal measures, or scale of widths, as we have hitherto done, an auxiliary line of measures, *o m*, is drawn, at such a distance behind the ground-line as to make *o c′* equal to G C. At *o* is erected a line of vertical measures, or scale of heights. Upon these lines the dimensions of width and height given in the plan and elevation are set off without change of scale. The dimensions of depth, given on the line G O, are set off at the same scale upon the ground-line, from the point *g*, and are transferred to the scale of depths, *g o*, by means of the point of half-distance, $D\frac{1}{2}$. If the scale were enlarged three times, a point of one-third-distance, $D\frac{1}{3}$, would be employed.

Fig. 97, *b*.

342. These three scales, or lines of measures, are what are called in geometry co-ordinate axes, and their point of meeting, *o*, is called the origin of co-ordinates, being the point from which the co-ordinates of any point are measured. These being laid off upon these axes, as explained in the previous paragraph, the position of any point in space is easily ascertained by drawing from the corresponding points on each axis lines parallel to the other axes, in each plane of pro-

jection; this gives the projection of the point in each of the three planes. Lines drawn from each of these points of projection, perpendicular to the plane in which it lies, and parallel to the other axis, will meet in space in a point which is the perspective required. As two of these axes are parallel to the picture, lines parallel to them are drawn parallel to their perspectives, while the third axis, being perpendicular to the picture, is, together with the lines parallel to it, directed to the centre, V^C. The perspective of the point 4, the vertex of the pyramid, is ascertained in this way, its distance from the origin, *o*, in each direction, being first marked upon the three axes, or lines of measures; its projection upon the three planes is then found; and finally the point itself is found where the three co-ordinates in space meet. The point 4 is seen to be the front upper right-hand corner of a parallelopiped, of which the origin, *o*, is the lower back left-hand corner.

343. But it is obvious that, as the intersection of two of these lines would suffice to determine the point, the projection of the point upon two planes is all that is required. The perspectives of the points 1, 2, and 3 are accordingly determined only upon the horizontal plane, and upon the vertical plane on the left, perpendicular to the picture.

It is worth while to point out that in the perspective plan the resemblance between the converging co-ordinates, and the edges of the pyramid above, is accidental. These lines would converge upon V^C, whatever the shape of the object to be drawn.

344. Fig. 97, *c*, shows that by omitting to sink the perspective plan these operations may all be conducted almost within the limits of the picture itself. It also shows that if we omit the letters and figures, and draw in only so much of the constructive lines as are actually necessary to determine points required, the work is simple and easy, and does not fill up the space devoted to the picture.

Fig. 97, *c*.

345. We have hitherto regarded the plane of the picture as a surface of indefinite extent, the exact limits of the picture itself being the last thing to be considered. In the system under consideration, however, the shape and size of the picture are determined beforehand, its lower limit being set at the ground-line, from which, on either side of the picture, arise vertical lines, which are its margins on the right and left. The points where they stand being noted on the orthographic plan, horizontal boundary-lines drawn through these points and the station point enclose a trapezoidal surface, which is the area seen in the picture. These horizontal boundaries, radiating from the station-point, have their perspectives vertical, in accordance with the nature of converging lines (231). Their perspectives are the vertical margins of the picture, which to the spectator at the station point seem to cover and coincide with the horizontal boundaries. All points, accordingly, situated on the horizontal boundaries of the trapezoidal area will appear in perspective upon the vertical margins of the picture. These ver-

Vertical margins.

tical margins are not necessarily set at equal distances from the centre, V^C.

346. If, for instance, the distance of the centre, V^C, from one of the vertical margins of the picture being determined, the distance of the station-point, S, in front of the picture, is three or four times that distance, then the point of one-third or one-quarter distance, $D^C\frac{1}{3}$ or $D^C\frac{1}{4}$, will fall exactly upon that vertical margin. If, also, any lines of the plan are prolonged until they cut the boundaries of the visible horizontal plane, these points of intersection will also appear in perspective upon the vertical margins of the picture. Such lines may accordingly be put into perspective without finding any points except just at the edges of the picture, and upon a line of depths.

347. Fig. 98 illustrates this procedure, showing how the same perspective plan as that determined in Fig. 97 can thus be obtained. At *a* we have, as before, a small-scale orthographic plan. Assuming $p\,p'$, in the picture plane, as the right and left-hand limits of the picture, diverging lines drawn through these points from the station-point, S, establish the horizontal boundaries of the trapezoidal area. If, now, the sides and diagonals of the plan of the pyramid be extended till they cut these boundaries on either side and the ground-line in front, we shall have the points numbered 1, 2, 3, 4, 5, 6, 7, and 8, their projections upon the line of depths, drawn at right angles to p, being figured 1′, 2′, 3′, 7′, 8′. If, now, we double the scale, as in the previous case, the points 1′, 2′, 3′, 7′, and 8′, are easily found

Fig. 98.

upon the scale of depths, as before, and 1, 2, 3, 7, and 8, at the same levels upon the vertical margins. The points 4′, 5′, and 6,′ being taken on the small-scale line of widths serve to determine the points 4, 5, and 6 upon the ground-line.

348. The perspectives of these eight points being thus ascertained, it is easy, by drawing the lines 1, 6; 2, 7; and 4, 8, to draw the perspective plan of the visible half of the pyramid. The point 9, the projection of its vertex, is got by drawing its projection upon the ground-line in the orthographic plan. Indeed, in practice, this method and that of co-ordinates are used interchangeably, as the conditions of each case may render advisable.

349. In Mr. Adhémar's treatise the trapezoidal figure, with its horizontal boundary-lines, is drawn in the orthographic plan, and the corresponding vertical margins are drawn at the edges of the picture, even in cases where the special device described in the previous paragraph is not employed. They serve to define the limits of the work, and it is often convenient, even when one is employing the ordinary method of co-ordinates with the origin of co-ordinates in the plane of the picture, to have the line of depths, *i. e.*, the line of perpendicular or normal measures, and the ground-line, or line of horizontal measures, meet in the line of vertical measures, which in that case coincides with the vertical margin in the corner of the picture. This point is then the origin of co-ordinates, as happens in Fig. 92. In employing the method of small-scale data also, although

the origin of co-ordinates is at a distance, it is often convenient to have the ground-line, the line of depths, and the vertical margin, meet at a point.

350. But that this is a mere matter of convenience, which involves no principle, and really has no effect upon the method of working out the perspective problem, may be seen in Figs. 99 and 100, which illustrate the relation between the vertical margins of a picture and the system of small-scale data, according to which they are made.

Vertical margins not necessary with small-scale data.

351. Fig. 99 shows an orthographic plan, on a small scale, with the station-point, S, the point *a*, of which the perspective is to be found, and the plane of the picture, at G L, at right angles to the Axis, with the centre of the picture at V^c, and the point of distance at D, V^c D being equal to S V^c. The line of depth, G O′ O″, touches the ground-line at G, and the left-hand boundary of the visible area is drawn from S through this point. But as this line plays no part in the solution of the problem its position is immaterial, and it might just as well have been directed more to the left, as shown by the dotted line. The right-hand boundary is also drawn at pleasure. Being drawn, they fix the width of the picture, G L.

Fig. 99, A and B.

Fig. 99, B, shows the same points in the side view, *p p* being the plane of the picture, *a* the point to be put into perspective, and *a g* the horizontal plane on which it lies, cutting the plane of the picture at the ground-line, *g*. $V^c g$, equal to *c a*, shows how far the point in question lies below the level of the eye at S.

352. Fig. 99, C, shows the picture itself on a scale three times that of the plan, V^C being the centre, and $D\frac{1}{3}$ the point of one-third distance.

Fig. 99, C.

The origin of co-ordinates, *o*, is found by setting off V^C *i*, equal to V^C *g* in Fig. 99 B, and *i o*, equal to V^C G in Fig. 99 A. The lines *o z*, *o e*, and *o g*, are then the three co-ordinate axes. The ground-line, *g l*, is three times as far from the Horizon as in Fig. 99, B.

353. As the scale employed in these operations is one-third of the scale employed in the picture, it follows that the plane of measures containing V^C *i*, *o* and *e* is three times as far from the eye as is the picture. In the orthographic plan it lies at O′. This distance, G O′, of which *o g* is the perspective, may be found directly by drawing a line from $D\frac{1}{3}$ through *o* until it intersects the ground-line at *o′*. The distance, *g o′*, gives the distance of *o* behind the picture, on the scale of the plan. It is equal to G O′. This gives the position of the plane of measures (93), the plane in which these dimensions are taken.

The plane of measures.

354. The horizontal co-ordinates of any point in the perspective plan can now be ascertained. Let *a* be such a point. Its horizontal co-ordinate, *u a*, equal to *o′ u′*, being laid off on the horizontal axis from *o*, gives the point *e* on that axis, and the perspective of the point *a* will lie in the normal perspective line drawn through *e* from the centre, V^C. The point *u* upon the line of depths, *o g*, may be ascertained either by laying off upon that line the perspective of the distance, O′ *u*, from *o*, or the distance, G *u*, from *g*. Either of these

distances being measured upon the ground-line, from g or from o', determines the point u', which may then be transferred to the line of depth, at u, by means of the point of distance, $D\frac{1}{3}$, A horizontal line from u will cut the normal line through e at the desired point, a.

If the desired point lay above the ground-plane, its height could be set off at the same scale upon a vertical ordinate erected at e, and a normal line drawn through V^C, and the point thus determined would cut an ordinate erected at a at the required point.

355. It will be noticed that all these operations are quite independent of the limits to be given to the picture, and would have gone on in exactly the same way if the vertical margin had been drawn to the left of the point g, just as the lower margin is independent of the ground-line. Moreover, most of these operations are quite independent of the *scale* to be employed,— that is to say, of the number of times the picture is to be magnified. This is determined solely by the distance at which the ground-line is set below the Horizon.

356. This is illustrated in Fig. 99, D, which is identical with Fig. 99, C, except that the line $g\ l$ is between five and six times as far from the Horizon as is the horizontal axis, $o\ e$, instead of three times. This gives upon the ground-line the distance, $g\ o''$, and upon the orthographic plan to point O'', for the position of the plane of measures.

Fig. 99, D.

The fractional point of distance remains unchanged, and the point *u′* is as far from *g* as before; *u* and *a* are then easily found. If the dimensions of the picture are enlarged proportionally, as is here done, *a* will be found to occupy the same relative position in the larger picture that it does in the smaller picture above.

It is not of course necessary to know how large a magnifying power one is employing. The ground-line, or the vertical margins, may be set wherever seems best. The rest will take care of itself.

357. Fig. 100 shows that the results attained are also quite independent of the size and scale of the picture shown in the orthographic plan. We have at A and B exactly the same conditions as in the previous figure, except that the plane of the picture is further from the spectator, and nearer the point *a*. The picture there indicated is accordingly larger, the point of distance further from the centre, and the distance, G *u*, taken on the line of depth, just so much smaller. This line, being still drawn to the edge of the picture, is further from the axis, and the dimension, G V^c, is greater than before.

Fig. 100, A and B.

In Fig. 100, C, however, we obtain exactly the same result as in Fig. 99, C, the scale of the enlarged drawing being the same.

Fig. 100, C.

In this figure the ground-line meets the left-hand vertical margin just in the corner of the picture. The original, at A, being larger than in the previous figure, does not need to be magnified as many times, hardly

more than twice, and the ground-line is accordingly set nearer the Horizon.

In Fig. 100, D, the same result is a third time attained, the line of depth not being brought to the corner of the picture, but set just as far from the axis as in Fig. 99, the point of distance, however, being the same as in Fig. 100, C. Fig. 100, D.

358. It appears from these examples that, if it were for any reason convenient to employ different systems of ground-lines and points of distance, or different lines of depth, in the same picture, one part of it being put in by one system and another by another, there would be no discrepancy in the results. We shall have occasion to avail ourselves of this before we get to the end of the chapter. (375.)

359. We have seen that the method of co-ordinates is specially useful in putting into perspective irregular figures, figures that have to be treated as a series of points. In drawing mouldings, for instance, especially when the scale of the drawing is so large that they have to be determined with great precision, the system of co-ordinates enables us to fix the horizontal and vertical position of as many points as we please, and thus to attain any degree of exactness in the profiling that we may desire. A single profile being ascertained, any length of moulding may be drawn by drawing lines through the points thus determined to the proper vanishing-points. Auxiliary directrices. Mouldings.

360. Fig. 101 shows how the horizontal and vertical dimensions that define the outline of a cornice may be determined in either of three ways. At *a* is shown the section of a cornice, with the horizontal and vertical co-ordinates of the principal points. At *b* this section is put into perspective by means of the perspective plan R above and the point of distance, D^R, and the parallel lines of the cornice are drawn through the points thus determined. At *c* the perspective of the cornice at the corner, "on the mitre," is ascertained in like manner, with the aid of the vanishing-point of 45°, V^X. This profile serves for the cornice on the right as well as for that on the left. At *d* the section parallel to the picture is ascertained, and employed for a cornice going off to the right. Being used parallel to the picture, it is put into perspective without change.

Fig. 101.

Mr. Adhémar employs all these profiles at once, as auxiliary to each other, thus dispensing with both vanishing-points, as may be seen at *e*. The relative position of these sections may be seen in the perspective plan at the top.

361. This device is of special service in putting circular mouldings into perspective, as may be seen in Fig. 102, which is borrowed, with the omission of the construction lines, from the plates of his treatise. The construction of these profiles is very easy, and as many of them can be employed as seems necessary.

Circular mouldings. Fig. 102.

362. In very large drawings the use of auxiliary profiles as directrices is of special advantage, because, even when the vanishing-points are within reach, the

necessary rulers and straight edges are so long and so flexible that it is difficult to keep the lines true and the mouldings properly proportioned throughout.

363. It has already been said that the reason why the method of the Perspective Plan has attained so little vogue, and has generally been set aside for the method of Direct Projection, is this: that the lines of the perspective plan generally meet at so acute an angle as to make it difficult to determine their exact point of intersection. This objection is overcome by the device of sinking the plan, so that the perspective lines may intersect more nearly at right angles. But however satisfactorily this may be arranged for the foreground of the plan, lines in the more distant parts must always grow gradually more nearly parallel to the horizon and to each other, so that in just that part of the drawing where the scale is the smallest, and precision, accordingly, most important, precision is the most difficult to obtain.

Remote objects.

364. To remedy this evil Mr. Adhémar proposes that the more distant objects, or the more distant portions of an object, shall be projected not upon a horizontal plane but upon an inclined plane, thus increasing the vertical distances and the angles at which the lines of the plan intersect with one another. Fig. 103 shows the device in question. At A is shown the station-point, S, the plane of the picture, $p\,p'$, seen in profile, and the elevation, or edge, of a horizontal ring tangent to the

The Inclined Perspective Plan.

Fig 101.

Plate XXII.

Fig. 103, A.

plane of the picture at *p*, and the projection of this ring upon the ground-plan at *g g*. This will of course be a circle, and its perspective, as seen from S, will be an ellipse, the major axis of which is horizontal and rather less than the diameter of the circle, and the minor axis vertical, and equal to the distance, *d g*. This ellipse is shown at *a*, where is shown also the ground-line, *g l*, the centre, C, the major axis of the ellipse at *f f*, and the horizontal diameter of the circle, which appears as a horizontal chord of the ellipse at *e e*, the further half of the circle appearing, of course, smaller than the nearer half. The centre of the circle is seen, in the plane of the picture, at *c*.

365. Mr. Adhémar's device for making the perspective of the further half of the circle as wide as that of the nearer half is shown at B. The further half of the ring is projected not upon a horizontal but upon an inclined plane. This plane is shown in the figure, cutting the plane of the picture in a new ground-line, at *g'*, the horizon of this plane being not at V^C, but at $V^{C'}$. Using this new horizon and this new ground-line, the projection of the further half of the ring can be drawn, as shown at *b*, with any desired precision, the same as upon any inclined plane.

Fig. 103, B.

But this can be equally well and more simply accomplished, merely by sinking to a still lower level the further half of the circle in which the ring is projected, that is to say, by projecting the further half of the ring not upon an inclined plane but upon a lower horizontal plane, as is done in Fig. 103, C. This,

Fig. 103, C.

to be sure, though it widens the projection of the further parts as much as may be desired, brings their perspective lower down than that of the nearer parts, as at *c*, which is awkward. But this can be got over, by not only sinking that portion of the perspective plan, but also simultaneously raising the station-point. If this is done judiciously, as at D, the two portions of the perspective plan will preserve their relative position, as at *d*. The next figure shows how this may be effected (370).

Fig. 103, D.

366. The results reached in B and D are substantially alike, and it is plain that sinking the remote portions of the perspective plan, instead of using an inclined plane of projection, accomplishes the practical end in view, by what seems to be a simpler method. It is worth while to point out, however, before leaving the subject, that the results in the two cases are not only similar but identical; that is to say that the perspective of the projection upon the sunken horizontal plane coincides with that of the projection upon the inclined plane, not only at its extremities but at every point.

The sunk plane preferable to the inclined plane.

367. This may be seen in Fig. 104, where the line A G, arbitrarily divided into two parts at the point B is seen projected both upon a horizontal plane at *a*, *b*, and *g*, and upon an inclined plane at *a′*, *b′*, and *g*. The figure shows that if the new station-point S′, immediately above S, is so taken that the perspective of *a* as seen from S coincides with that of *a′* as seen from S, the point *g* common to both projec-

The results identical. Fig. 104.

tions being its own perspective, then the perspective of b as seen from S′ will coincide with that of b' as seen from S. That is to say, if the perspectives of the two projections coincide at their extreme points they will also coincide at every intermediate point.

368. The proof of this follows from the relations of the lines of the figure, in which perpendiculars let fall from the two station-points S and S′ give the corresponding Horizons at V^C, and $V^{C'}$. For, in the first place, since the trapezoids SS′ C′ a'' and $g\ a\ a'\ a''$ are divided by the line a S′ into similar triangles, the trapezoids themselves must be similar, and since their homologous sides are parallel, their diagonals S $V^{C'}$ and $g\ a'$ are also parallel. The right-angled triangles S S′ $V^{C'}$ and $b\ b'\ g$, having their vertices upon the line C′ g are accordingly similar, and their heights, S′ $V^{C'}$ and $b\ g$, are proportional to their bases S S′ and $b\ b'$. But the scalene triangles S S′ b'' and $b\ b'\ b''$ are also similar, and their heights also are proportional to their bases S S′ and $b\,b'$, which are the same as those of the right-angled triangles. Their heights must accordingly be the same as the heights of the right-angled triangles, and their vertices at b'' must also fall upon the line $V^{C'}$ g. That is to say, the perspective of b' as seen from S, coincides with that of b as seen from S′, and this is true wherever the transverse line $b\ b'$ may be drawn.

369. Since, as has been shown, S $V^{C'}$ and $g\ a'$ are parallel whenever S a' and S′ a intersect in the plane of the picture, it follows that the point $V^{C'}$ which marks the position of the Horizon of the horizontal plane of

projection $a\ b\ g$, when the eye is at S', marks also the horizon of the inclined plane of projection $a'\ b'\ g$ when the eye is at S, and that g marks the ground-line of both the sunk plane and the inclined plane.

370. This enables us to answer a question raised in regard to Fig. 103, D and d (367), and to say just how far the Horizon must be raised and the ground-line lowered in order to make the perspectives of successive sunk planes continuous, one with another, like the perspectives of successive inclined planes. The successive horizons and ground-lines of the horizontal planes are to be drawn just where the successive horizons and ground-lines of the inclined planes would be, the distance that the station-point is raised being proportional to its distance in front of the picture, and the distance that the horizontal plane is lowered being proportional to its distance behind the picture.

To find C′ and g′. Fig. 103, D.

371. The identity of the results obtained by these two processes is again illustrated in Fig. 105, a, b, c, and d. At a is the orthographic projection of a circle, tangent to the plane of the picture, G L. The area of this circle is divided into four segments by planes which cut the line of depth at the points A, B, D, E, and G. In the Fig. 105, c, above, these segments are shown projected upon four horizontal planes having their ground-lines at g, g'' g''', and g^{IV}, the corresponding station-points, S, S'', S''', and S^{IV}, being so adjusted, in

The results the same by either method. Fig. 105, a.

Fig. 105, c. Successive horizontal planes.

accordance with the relations shown in Fig. 104, that the perspectives of the plans of the different segments form a continuous figure, the perspective of the further edge of each segment, as seen from the lower station-point, coinciding with that of the hither edge of the segment below, as seen from the station-point next above.

372. Alongside, to the right, drawn to the same scale, is the perspective plan of the circle. The left side shows the perspective of its projection upon the ground-plane, *a g*. This is an ellipse, divided into the four segments corresponding to those into which the plan is divided. The right-hand side shows the perspectives of the projections upon the three sunken planes, which have their ground-lines at g'', g''', and g^{IV}, as they appear when viewed, respectively, from S'', S''', and S^{IV}. The second segment of the second ellipse, the third of the third, and the fourth of the fourth, are extended so as to be as wide as the first segment of the first, and, like it, are drawn in with a full line, thus making a continuous figure, in which the remoter parts of the circular surface are as advantageously displayed as the nearer parts.

Fig. 105, *b*. Successive inclined planes.

373. In Fig. 105, *b*, we see that exactly the same result is reached if, instead of thus using successive horizontal planes, we employ Mr. Adhémar's device of successive inclined planes. Not only in this case do the perspectives themselves coincide at all points, but the ground-lines and horizons of the inclined planes are identical with those of the

horizontal planes in the previous figure. The practical work of thus constructing the perspective drawing is then the same, whichever method we prefer to employ. In either, the dimensions of width and height may be laid off, according to the scale of the picture, upon the successive ground-lines and upon the vertical margins. The dimension of depth in each plane may also be laid off upon the same ground-line, and transferred to the line of depths by the successive points of distance, which, upon each Horizon, will give the common distance of the successive station-points from the picture.

374. This is illustrated in the right-hand side of Fig. 105, *d*, which shows the successive perspective plans figured in Fig. 105, *b*, upon a scale double that of the previous figures. In two of the four segments shown in plan in Fig. 105, *a*, a point is taken at random, and its co-ordinates, measured from the ground-line and from a line of depths erected at L, are shown. These points, numbered 1 and 4, in the first and fourth planes, are put into perspective by means of the successive ground-lines $g\,l$ and $g^{\text{IV}}\,l^{\text{IV}}$, and the points of half-distance, $D\frac{1}{2}$ and $D\frac{1}{2}^{\text{IV}}$. Any number of points of the circle, or of any other figure, could be put into perspective upon the successive planes in the same way, just as in Fig. 92, Plate XX. (330). **Fig. 105, *d*.**

375. These figures show also that if, in either figure, 105, *b*, or *c*, we consider the vertical planes which divide the plan into segments to be so many successive planes of measures (93) intersecting the four successive ground-planes, whether horizontal **The results identical.**

or inclined, in the ground-lines g_2l_2, g_3l_3, g_4l_4, the perspectives of the successive segments will form a consecutive series. Whether viewed from S in Fig. 105, *b*, or from the successive station-points of 105, *c*, they will present the appearance of Fig. 105, *d*. This is an additional illustration of the point made in a preceding paragraph (358), and illustrated in Figs. 99 and 100 of the previous plate, that when a perspective drawing is enlarged to a given scale it makes no difference in the result at what distance from the station-point the plane of measures in the orthographic plan is set, and that, if convenient, part of a picture may be put in by means of one such plane, with its corresponding ground-line, line of depths, and point of distance, and part by means of another.

Fig. 105, *d*. Small-scale data.

376. This also is illustrated in Fig. 105, *d*, in which the methods of Figs. 99 and 100 are applied to produce exactly the same outline upon the left-hand side of the figure by the method of small-scale data as has been produced on the right-hand side by the method of co-ordinates (374). The figure shows the successive ground-lines, $g\,l$, g_2l_2, g_3l_3, g_4l_4, and the successive lines of depth, gC, g_2C$''$, g_3C$'''$, and g_4C$^{\text{IV}}$. The successive origins of co-ordinates, o, o_2, o_3, and o_4, are determined as in Figs. 99 and 100, and the small-scale data taken from the orthographic plan are laid off from them by aid of the successive fractional points of distance, D_1, D_2, D_3, and D_4. The co-ordinates of the points 1, 2, 3, and 4, on the left-hand side of the circle, corresponding to those already found on the right-

hand side, are thus used in the figure, and their perspectives found.

377. This process, also, is accordingly equally available, and is identical in its methods and results, whether we suppose the successive ground-planes to be inclined, as Mr. Adhémar has them, or to be horizontal, as we have suggested, and as seems rather more in harmony with the conceptions of this treatise, and with the habitual use of the sunk perspective plan to effect the same ends.

378. We have seen, in Fig. 103, *c* (365), that if we constantly lower the ground-plane so as to get a better view of its more remote parts, we must at the same time raise the station-point and the Horizon. Otherwise, the perspective plan will be dislocated, as shown in Fig. 101, *d*.

In practice the Horizon need not be raised.

But, in practice, this stretching out of the more remote portion of a plan is generally needed not in its whole extent, but only in part, often only for particular objects. Such isolated details are much more simply and conveniently treated by the use of the ordinary sunk horizontal plan, without moving the station-point. The advantage of this method in getting such auxiliary operations out of the way of the picture has frequently been illustrated in these pages; as, for example, in Fig. 20, Plate VII.

379. Figs. 106, 107, 108, and 109, Plate XXIII., show the application of this method to the interior of the *Halle aux Blés*, in Paris. The main lines of these figures are taken from Mr. Adhémar's treatise, though the construction lines are some-

Plate XXIII. Figs. 106, 107, 108, and 109.

what differently adjusted, and the notation altered. It is hardly necessary, after what has been said, to enter into a detailed explanation of these examples. They reproduce the main features of Figs. 99 and 100, as well as of Figs. 103 and 105. Two points, a and b, in the second and third planes respectively, are ascertained by the use of small-scale data.

CHAPTER XVI.

THE INVERSE PROCESS.

380. It is sometimes desirable to invert the procedure described in the preceding chapters. Instead of beginning with an orthographic plan and elevation, deriving thence a perspective plan and perhaps, as in the previous chapter, a perspective elevation, and then finally arriving at a complete perspective drawing, it is often possible by a reverse process to derive the perspective plan and elevation from the drawing, and from them to obtain the actual shape of the object, and its relations to the spectator and the plane of the picture. Its dimensions can also be determined if the dimensions of certain lines in it are known.

Given the perspective to find the real form and dimensions.

To effect this with any approach to precision it is necessary that the perspective lines shall make a sufficiently large angle with each other or with the Horizon clearly to indicate the position of the vanishing-points; that is to say, the object shown must either be large, near at hand, or considerably above or below the eye.

381. If the object is in oblique, or three-point perspective, and its three vanishing-points can be fixed with precision, there is no difficulty, as has been shown in Chapter VII., in determining the position of the spectator. This fixes the centre of the picture,

Oblique, or three-point perspective.

the station-point, the distance of the station-point from the centre and from each of the vanishing-points, and all the points of distance. For lines connecting the three vanishing-points represent the three horizons, and the meeting-point of the perpendiculars let fall from the angles of the triangle thus formed upon the opposite sides, is the centre, V^C; the distance of the station-point in front of this point, and its distance from each of the vanishing-points, is then easily determined (168), and the points of distance found. If, then, the length of any of the vanishing lines is known, a line of measures parallel to one of the horizons can be drawn through one of its extremities, its true dimension according to the scale of the drawing, or of that part of it, found by means of a point of distance, and the scale of the drawing and the dimensions of every other part ascertained.

382. This is illustrated in Plate XXIV., Fig. 110. If
Plate XXIV. Fig. 110. we suppose the perspective of the rectangular block to be given, the vanishing-points V^L, V^M, and V^O, and the traces that connect them, can be obtained, the ground-lines, or lines of measures *o m, l m*, and, if necessary, *l o*, drawn, and the relative dimensions of the edges determined. Whether the block is large or small cannot of course be learned. The size of the miniature block, supposed to be in contact with the picture at *a*, is determined, its edges being equal to *a l*, *a m*, and *a o;* but there is no means of knowing how much larger the block itself is than this miniature representative. To determine this we must know either the actual dimensions of one edge of the block, or the

distance of the block behind the picture. Neither of these can be shown by the picture itself.

383. If the object is drawn in two-point, or angular perspective, as is generally the case, it does not suffice for the determination of its shape, position, and relations to the spectator that the vanishing-points of its principal lines should be known. For fixing two vanishing-points does not, as fixing three does, determine the position of the spectator and of the centre of the picture, and thence of the points of distance, nor does it determine the *attitude* of the object, or the angles its sides make with the plane of the picture. Fixing the vanishing-points only restricts the *locus* of the spectator's position to the semicircle subtended by the line joining them; they determine neither the attitude of the object nor its shape. In Fig. 111, for instance, we have at A and B the same perspective and the same vanishing-points. But at A the station-point, S, and the centre, V^C, are assumed to be well over towards the right, and at B towards the left. The perspective plans and the elevations derived from them are shown below. The plans are alike, but the points of distance being different the dimensions found upon the ground-lines are different, and the proportions of the building and the slope of the roof come out differently. But while the buildings, though differing in size and shape, are alike in perspective, the doors and windows, which are of the same size and shape in one building as in the other, come out differently in perspective.

Two-point, or angular perspective.

Fig. 111.

The centre, V^C, given.

384. In order to interpret correctly a drawing made in angular, or two-point perspective, it is necessary to have definite information as to the position either of the centre, V^C, of the vanishing-point at 45°, V^X, or of one of the points of distance, D^R or D^L. The centre is generally nearly in the middle of the picture, but that it is exactly there is not to be taken for granted. Its position is often precisely indicated, however, by some secondary object, which is drawn in parallel perspective; and it is always a good plan to introduce some such object as the pile of boards in Fig. 111, A, as a guide to the spectator.

The vanishing-point of 45° given to find S and V^C. Fig. 112.

385. It often happens, however, especially in architectural drawings, that the nature of the subject is such as to furnish diagonal lines lying at 45° with the principal directions, lines that we have called X dividing the angle made by the lines R and L. Fig. 112 shows, by means of a little elementary geometry, how in this case the station-point, S, is to be found, V^R, V^L, and V^X being given. As the angle $V^R S V^X$ is an inscribed angle of 45°, its sides must include an arc of 90°. A line drawn from x in the figure through V^X to the opposite circumference fixes the position of S, and hence of V^C, D^R, and D^L.

Some real dimensions being known, to find a point of distance.

386. It is not often that the position of either of the points of distance can be detected by mere inspection of the picture. But it often happens that the real or proportional dimensions of some of the lines in the picture are known. In that case one of the points of distance can be ascer-

tained, and the other elements of the problem then easily determined.

Let us suppose, for instance, in Fig. 113, that the rise and tread of the steps are known to be six and twelve inches. A line of equal measures, $l\,r$, laid off parallel to the Horizon, from the front edge of the first step, in length equal to twice the vertical edge, forms, with the horizontal line in perspective, and a third line, joining their further ends, the three sides of an isosceles triangle. The vanishing-point of the third line, the base of the triangle, gives the point of distance, D^R. The distance from this point to its corresponding vanishing-point is the distance of the station-point from that vanishing-point; that is to say, $D^R\,V^R = V^R S$. S, which must lie somewhere in the semicircle of which $V^L\,V^R$ is the diameter, is then easily found, as in the figure. V^C, D^L, and V^X, immediately follow.

The point of distance being given to find S, C, etc. Fig. 113.

387. When the drawing to be interpreted is made in parallel perspective it is generally easy enough to find the centre of the picture, the vanishing-point of the lines perpendicular to it. But, as in the previous case, it is impossible to tell what is the real shape of the objects represented, or to know from what distance the picture should be looked at, unless the real shape of some one of the objects is known independently. If Fig. 20, for instance, Plate VI., is looked at from a point about three inches in front of V^C, as may be done by looking at it through a pin-hole, so as to

One-point, or parallel perspective.

obtain a clear image on the retina, the little pavilion represented looks about square, the steps on the side seeming very steep. Seen from a distance of several feet it looks two or three times as long as it is wide, and the steps seem of very easy grade. The posts at the corners are presumably square, and the lines of the pavement and of the hips of the roof, in plan, are presumably directed to the vanishing-point of 45°, which is the point of distance; and the steps have presumably the same slope as those on the right. The point of distance can be found by the same method as in the preceding paragraph, and the true shape of all the objects in the picture determined.

Acute and obtuse angles.
Fig 114.

388. Of course these results are based upon the understanding that the objects represented are rectangular. If the lines that define them are known to form acute or obtuse angles, instead of angles of 90°, the line joining their vanishing-points must be treated as the chord of a circle instead of as a diameter, as is done in Fig. 114.

Fig. 114, A.

At A is shown an obelisk in perspective, presumably square in plan. The methods described in the previous paragraphs suffice to determine successively the principal vanishing-points, V^R, V^L, and V^X, the centre, V^C, the points of distance, D^R and D^L, and a perspective plan. The dimensions can then be determined, according to the scale of the drawing, and that scale may be determined if any one of the dimensions is known.

At Fig. 114, B, is another drawing, the horizontal and vertical lines of which are identical with the first. But this obelisk is understood to be triangular and equilateral, with angles of 60° instead of 90°. A perspective plan, with the vanishing-points V^R and V^L, are easily determined, as is also V^X, the vanishing-point of the line bisecting the solid angle in contact with the picture. These elements suffice to determine the orthographic plan, Fig. 114, C. As the angle at the station-point, S, is only 60°, in place of 90°, it is included in an arc of 240°, the point S lying somewhere in that arc, which is its *locus*. The point V^X, however, enables us, as the point $V^{X'}$ did in the previous case, to fix the exact position of S, by drawing a line through the summit of the arc at X, and the point V^X. If then the eye is placed in front of Fig. 114, A, opposite $V^{C'}$, at the distance indicated by S′, on the plan below, the obelisk will look square; if it is placed opposite V^C, in Fig. 114, B, at the distance indicated by S, it will appear to have the section of an equilateral triangle.

Fig. 114, B.

Fig. 114, C.

389. The little pyramid on top is, however, differently drawn in the two cases, and the position of the apex suffices to show that the upper figure has four sides, the lower but three. The angles at which these sides meet, however, is necessarily intermediate.

390. The fact that acute or obtuse angles can thus be interpreted as right angles makes it difficult to represent them satisfactorily when there is nothing else in the picture to guide the judgment. It often happens in

the case of buildings situated where two streets meet at an odd angle that drawings of them look as if the buildings were square. To obviate this it is necessary, as has been said, to introduce something which is unmistakably rectangular, such as an awning, or chimney, or a cart backing up to the sidewalk, like the pile of boards in Fig. 111, A.

391. If it were not, indeed, for the facility with which the mind thus gives the most reasonable interpretation to the phenomena that meet the eye, even the right angles shown in perspective would generally look either acute or obtuse, since it is only when the eye is exactly at the station-point, or rather when it is on the circumference of a semicircle lying between the vanishing-points, that the angle really looks as a right angle would. But these distortions, like the other distortions that arise from abandoning the station-point, are made light of by the intelligence. It is only in the case of circles, cylinders, and spheres that one is disturbed by them. In those cases, indeed, remaining at the station-point hardly suffices to reconcile one to the drawing, as has been explained in Chapter XII.

CHAPTER XVII.

SUMMARY. — PRINCIPLES.

392. Let us now review the ground gone over in the previous chapters, giving the subject a somewhat more formal treatment. We can then proceed to extend the principles and methods employed to the solution of some problems which we have not as yet encountered.

Propositions and Definitions.

393. A perspective drawing is a representation of the lines that would result from the projection upon a transparent plane of a plane or solid figure situated behind that plane, the lines of projection being visual rays converging from every point of the figure to the eye of a spectator situated in front of the plane.

This sort of projection is called Conical Projection, as distinguished from Orthographic Projection, in which the lines of projection are parallel, and are normal to the plane of projection. See Fig. 115, Plate XXV.

Plate XXV., Fig. 115.

The transparent plane of projection is called the Perspective Plane.

The perspective plane.

The position of the spectator is called the Station-point.

The station-point, centre, and axis.

The point in the perspective-plane opposite the station-point is called the Centre. It is the orthographic projection of the station-point upon the perspective-plane, and is the point in that plane nearest the spectator's eye. It is not necessarily in the middle of the picture.

The line of projection drawn from the station-point normal to the perspective-plane and passing through the Centre, is called the Axis. Its length gives the distance of the eye in front of the perspective-plane.

Unless otherwise expressly mentioned, the perspective-plane is understood to be vertical, the Axis, accordingly, horizontal, and the Centre on a level with the eye at the station-point. The eye is supposed always to remain at the station-point.

394. The conical projection of a figure, line, or point upon the perspective-plane is called its perspective. When viewed from the station-point the perspective exactly coincides with and covers the object itself.

The perspective representation.

An object and its perspective are, accordingly, not to be distinguished in the picture; but it is necessary to distinguish between them in idea, and sometimes, to prevent confusion to speak of the *real* horizon.

Objects lying at the same distance as the perspective-plane are represented as of their real size. All lines and figures lying in the perspective plane are their own perspectives.

Scale dependent on position relative to the perspective-plane.

Objects behind the perspective-plane have their perspectives smaller than themselves. Objects in front of the perspective-plane may be conically projected back upon it, and have their perspectives larger than themselves. All objects have their remoter parts drawn to a constantly smaller scale than their nearer parts.

395. The surface upon which the drawing is made is called the plane of the paper, or plane of the picture. See Fig. 116, A.

The plane of the picture. Fig. 116.

When the objects to be drawn are very small, or the picture very large, the plane of the picture and the perspective-plane may coincide, as in Fig. 115.

But generally the objects to be drawn are many feet in dimension, as at B, Fig. 116, while the drawing is to be measured by inches; and as it is always convenient, for practical reasons, to have the plane of projection as near the object as possible, and consequently of about the same size, while the picture must be small and near at hand, the plane of the picture and the perspective-plane are generally nearly as far from each other as the spectator is from the object, as in Fig. 116.

In this case the drawing is a miniature, or small-scale representation, of the perspective lines supposed to be traced upon the perspective-plane. It may be regarded as a perspective of the perspective-plane, as if made upon a second transparent plane, near the eye, parallel to the perspective-plane. All the lines in the perspective-plane will be represented unchanged in direction, and reduced in a uniform ratio. The scale of

Scale.

the drawing will depend upon the relative distance of the two planes from the eye (94).

396. The same result may be reached by conceiving the drawing to be made not from the object itself, many feet distant and many feet in dimension, but from a miniature, or model, close to the plane of the picture, or even in contact with it. Any line that lies in the plane of the picture is then its own perspective (394). See Fig. 117.

In this case the perspective-plane and the plane of the picture coincide, as in the case of small objects. As this is the most convenient working hypothesis it is generally adopted, and the distinction between the two planes is not recognized.

In the previous chapters the perspective-plane, where it has been necessary to speak of it, has been called the plane of measures (93). In this chapter and the following one the perspective-plane and the plane of the picture will, as usual, be considered identical.

Planes.

Finite and infinite planes. Fig. 118.

397. Every plane figure, whether horizontal, vertical, or inclined, is regarded as lying in and forming a part of an indefinitely extended or infinite plane, which in its hither, or nearer, part intersects the perspective-plane in a line called its initial-line or trace. In its further part it tends to reach, but can never pass, an infinitely distant line, which is a great circle of the infinite sphere of which the station-point is the centre, and which

is called the vanishing-line, or horizon, of the plane. See Fig. 118.

The perspective of such an infinite plane extends, in the perspective plane, from the initial-line, or trace, which is its own perspective, since it lies in the perspective-plane (394), to the perspective of its horizon, but not beyond it. The perspective of the horizon of a plane is also called the horizon of the plane.

The initial-line, or trace, and horizon.

398. Planes that are parallel to one another are said to belong to the same system of planes. They have the same horizon, which is called the horizon of the system, but not the same traces. The initial-lines, or traces, of parallel planes, which are the lines in which they cut the plane of the picture, are naturally parallel to one another. A plane drawn through the station-point, in front of the picture, and through the horizon of the system, is a member of the system seen edgewise. It intersects the perspective-plane in the horizon of the system. The initial-line, or trace, of this plane and the perspective of its horizon coincide. See Fig. 119. Such a plane is called an Optical plane.

Parallel planes. Fig. 119.

The trace of a plane is accordingly parallel to its horizon.

399. Hence the line in which a plane passing through the station-point, and parallel to a given plane, intersects the perspective-plane, is the horizon of the given plane and of the system of planes to which it belongs. Such a plane, since it passes through the eye, is seen as a line, covering and coinciding with its horizon and the per-

spective of its horizon. Both the horizon of a plane and the perspective of this horizon may accordingly be found by *looking* in directions parallel to this plane.

400. Hence, whatever angle two planes, or two systems of planes, make with each other in space, the optical planes drawn through the station-point to their horizons, or to the lines which are the perspective of their horizons, will make the same angle at the station-point, in the air in front of the picture, as do the planes themselves. See Fig. 120.

The angle of two planes. Fig. 120.

401. A right line lying in a plane is called an element of that plane.

A right line normal or perpendicular to a plane is called an axis of that plane or system of planes.

All planes have an element, or system of elements, parallel to the picture.

All planes have an element, or system of elements, that is horizontal.

402. Planes parallel to the picture have all their elements parallel to the picture. See Fig. 121. The axes of such planes are normal to the picture and parallel to the Axis. The traces of such planes, and their horizons, are at an infinite distance, and cannot accordingly be represented. Lines and figures in any such plane are drawn at a uniform scale.

Planes parallel to the picture. Fig. 121.

All the planes of a system not parallel to the picture, whether horizontal, vertical, or oblique, have the same horizon but not the same traces.

Planes not parallel to the picture. Fig. 119.

Their initial-lines, or traces, are parallel to each other and to the horizon of the system, as above. See Fig. 119.

Planes that have elements that are perpendicular or normal to the picture are called normal planes. Their axes are parallel to the picture. The horizons of such planes pass through the centre of the picture. Both the horizons and their traces are perpendicular to their axes, and give the true slope and direction of the steepest lines of such planes; that is to say, of the planes themselves. These steepest lines are parallel to the picture. See Fig. 122.

Normal planes. Fig. 122.

The horizontal elements of normal planes are normal to the picture.

403. Horizontal planes are normal planes in which all the elements are horizontal. Their axes are vertical, and their horizon and traces are horizontal. The horizon of horizontal planes is called *the* Horizon. The most important horizontal plane is called the ground-plane, and its trace the ground-line. See Fig. 115.

Horizontal planes. Fig. 115.

Other planes besides horizontal planes have only one set of elements horizontal. If this is parallel to the picture, their traces and horizons will be horizontal; *i. e.*, parallel to the Horizon. These are sometimes called inclined planes. Their axes are inclined also, and lie in normal vertical planes. See Fig. 123.

Inclined planes. Fig. 123.

Other planes, besides those parallel to the picture, have only one set of elements parallel to the picture. If this is vertical they are called vertical planes. Their traces and horizons are ver-

Vertical planes.

tical, and their axes are horizontal, as shown at A, in Fig. 122.

Oblique planes. Fig. 124.

404. In general, those elements of planes that are parallel to the picture are neither vertical nor horizontal, and their horizontal elements are neither parallel nor normal to the picture. Such planes are called oblique planes. Their traces and horizons make an angle with the Horizon and the ground-line, but this angle is less than their true slope. Their axes also are oblique, being inclined both to the picture and to the ground-plane. See Fig. 124.

Lines.

Finite and infinite lines. Fig. 125.

405. Every line is regarded as lying in and forming part of an indefinitely extended or infinite line, which, if not parallel to the perspective-plane, or plane of the picture, pierces it at its hither, or nearer end, at a point called its initial-point, and at its further end tends to reach, but can never pass, an infinitely distant point, called its vanishing-point. See Fig. 125.

The initial-point, vanishing-point, and its perspective.

The perspective of such an indefinite line is drawn in the plane of the picture from the initial-point, which is its own perspective, towards, or even up to, the perspective of its vanishing-point, but it can never pass beyond this point. It is the line in which the plane of rays, drawn from the given line to the station-point, intersects the perspective-plane.

Parallel lines. Fig. 126.

406. Lines that are parallel to one another, whether horizontal, vertical, or inclined, are said to belong to the same system of lines. They

have the same vanishing-point, which is called the vanishing-point of the system. A line from the station-point to this vanishing-point is a member of the system, seen endwise. It pierces the perspective-plane in the perspective of the vanishing-point. Its initial-point and the perspective of its vanishing-point coincide. See Fig. 126. The portion of this line lying in front of the picture, between the station-point and the perspective of the vanishing-point, is called an Optical line.

407. Hence, the point at which a line drawn through the station-point, parallel to a given line, pierces the perspective-plane, is the perspective of the vanishing-point of the given line, and of the system to which the line belongs. Such a line, since it passes through the eye, appears as a point, covering and coinciding with the vanishing-point and its perspective. The vanishing-point of a given line, and the perspective of this vanishing-point, may accordingly be found by *looking* in a direction parallel to that of the line.

The angle made by two lines. Fig. 127.

408. Hence, whatever angle two lines make with one another in space, the optical lines drawn from the station-point, in front of the picture, to the perspective of their vanishing-points, will make the same angle in the air at the station-point. See Fig. 127.

Lines parallel to the picture.

409. Lines belonging to a system that is parallel to the perspective-plane, whether vertical, horizontal, or inclined, have their perspectives parallel to themselves and to each other, their initial-points and the perspectives of their vanishing-point being alike at an infinite distance upon the perspective-plane. They are drawn to the same scale in all their parts. See Fig. 121.

Such are the axes of normal planes. If such planes are horizontal their axes are vertical, and have their vanishing-points in the zenith and nadir; if vertical, their axes are horizontal; if inclined, their axes are inclined, in a contrary sense. See Fig. 122.

Lines not parallel to the picture.

410. The perspectives of a system of lines not parallel to the perspective-plane, whether horizontal or inclined, are not parallel to the lines themselves nor to each other, but converge towards the perspective of their vanishing-point. They are drawn to a constantly diminishing scale in their remoter parts. See Fig. 126.

Such are the axes of oblique planes, which are oblique also, being inclined both to the picture and to the ground plane. Such, also, are the axes of inclined planes which have their horizontal element parallel to the picture. Their axes also are inclined, in the contrary sense, and have their vanishing-points in a vertical line passing through the Centre. The axes of all vertical planes have their vanishing-points in the Horizon.

Normal lines. Fig. 128.

411. Lines normal to the perspective-plane, and parallel to the axis, have the centre of the picture for their vanishing-point. See Fig. 128.

Such are the axes of planes parallel to the picture. See Fig. 121.

Lines in the plane of the picture.

Lines that lie in the perspective-plane are their own perspectives (394), and are consequently drawn of their own size and shape. See Fig. 117.

412. All lines lying in or parallel to a plane have their vanishing-points in the horizon of the plane, and the lines that lie in it have their initial-points in its initial-line, or trace. See Fig. 129. **Lines in planes. Fig. 129.**

[All the elements of the ground-plane have their vanishing-points in the Horizon and their initial-points in the ground-line. See Fig. 115.]

Conversely: — The horizon of a plane, passes through the vanishing-points of all the lines that lie in or are parallel to it, and the initial-line or trace through the initial-points of the lines that lie in it.

[Since two lines are sufficient to determine a plane, this proposition takes, in practice, the following form: —

The horizon of a plane, passes through the vanishing-points of any *two* lines that lie in it, and its initial line or trace through their initial-points.]

Hence: — The line of intersection of two planes has its initial-point at the intersection of their traces, and its vanishing-point at the intersection of their horizons. See Fig. 130. **A line in two planes. Fig. 130.**

Conversely: — The horizons of all the planes that can be passed through or parallel to a given line will intersect each other at its vanishing-point, and the traces of all planes passed through it will intersect each other at its initial-point.

413. Since a plane may be conceived to be passed through a line in any direction, and its trace and its

horizon may accordingly pass through its initial-point and its vanishing-point in any direction, it follows that any two parallel lines in the perspective-plane, drawn at random through the initial-point and the vanishing-point of a given line, may be taken to represent the initial-line, or trace, and the vanishing-line, or horizon, of a plane passing through the given line.

Accidental traces and initial-lines.

Hence : — Any line in a drawing that happens to pass through the initial-point of a perspective line may be taken as the trace of a plane containing the perspective line, a plane of which the horizon may be found by drawing a parallel line through the vanishing-point of the perspective line; *conversely*, any line that passes through the vanishing-point may be taken as the horizon of a plane, and the corresponding trace may be drawn parallel to it through the initial-point.

414. The perspective of the horizon, and the trace, or initial-line, of a plane are, as has been shown, always parallel to each other, and to the initial-lines of all the other planes of the system (398).

Lines parallel to the trace of a plane.

They are parallel also to the lines in which the plane in question, and all the other planes of the system, intersect all planes parallel to the perspective-plane and to the perspectives of these lines. For the lines in which the two systems of planes intersect are all parallel, and parallel to the picture, and their perspectives

are accordingly parallel to themselves and to each other. They are parallel also to all other lines lying in any of the planes of the systems, which are parallel to the perspective-plane and to their perspectives.

Points.

415. A given point is regarded as lying in a given line at a known distance from its initial-point. See Fig. 131.

Points in lines. Fig. 131.

In this case the position of the point may be considered to be given by polar co-ordinates, the initial-point being the origin of co-ordinates, and the direction being given by the vanishing-point.

When the given line is normal to the perspective-plane the initial-point is the orthographic projection of the point upon that plane. In this case the position of the point may be considered to be given by three rectangular co-ordinates, of which the co-ordinates of the initial-point, which lie at right angles to one another in the perspective-plane, are two, and the given line, lying at right angles to them both, the third. See Fig. 132.

Co-ordinates of a point. Fig. 132.

416. The perspective of a point is the point at which a ray drawn from the given point to the eye pierces the perspective-plane.

A point lying at the intersection of two lines has its perspective at the intersection of their perspectives. See Fig. 129, *a*.

A point in two lines, in the perspective-plane.

A point lying in the perspective-plane is its own perspective. See Fig. 129, *b*.

The Points of Distance.

417. Through a line, the perspective of which is given by its initial-point and the perspective of its vanishing-point, let an auxiliary plane be passed, its trace and horizon being drawn parallel to each other, in any convenient direction (413). See Fig. 133. They will pass through the initial-point and the perspective of the vanishing-point of the given line. Let now equal distances be assumed upon this initial-line, as at *m*, and upon the given line itself, — not its perspective, — as at *a*, measuring from the initial-point. The lines thus set off will form the two sides of an isosceles triangle I *m a*, lying in the auxiliary plane, behind the plane of the picture, having one of its sides, I *m*, in the plane of the picture. If now this triangle be completed its base will be a line also lying in the auxiliary plane, and the vanishing-point of this base will lie in the horizon of this plane. The perspective of this vanishing-point will be found in the horizon of the plane, as at D, in the figure. This vanishing-point, which is also the vanishing-point of all lines drawn parallel to the base of the isosceles triangle, and consequently intercepting equal portions upon the given line and the initial-line, is called a point of distance of the given line.

The point of distance. Fig. 133.

418. The perspective of a point of distance of any line is a point hardly less important than is the perspective of its vanishing-point. The latter serves to determine the direction in which to draw the perspective of a given line; the former serves to deter

Its perspective.

mine its length. For if, in the plane of the picture, any given length, as I *m*, be laid off upon the initial-line of the auxiliary plane, and a line be drawn *across* the perspective of the given line towards the perspective of the point of distance, this line will be the perspective of the base of the isosceles triangle, of which the length, I *m*, taken upon the initial-line is one side, and the length intercepted upon the given perspective-line is the perspective of the other side. An indefinite line, then, being given by its perspective, any required length can be determined upon it by setting off this length upon such an initial-line, or trace, and transferring it to the perspective of the given line by means of the perspective of a point of distance.

The isosceles triangle behind the picture.

419. The perspective of the point of distance will lie in the horizon of the auxiliary plane, as at D, as will also the perspective of the vanishing-point, as at V. These two points, which lie in the plane of the picture, and the station-point, S, which is in the air in front of the picture, form the vertices of a triangle lying in front of the plane of the picture, of which one side, V D, lies in that plane and the other two meet at the spectator's eye, or the station-point. One side of this triangle lies in the horizon of the auxiliary plane, and is accordingly parallel to that side of the isosceles triangle, behind the picture, which lies in the initial-line of the auxiliary plane; *i. e.*, V D is parallel to I *m;* the side that extends from the station-point to the perspective of the vanishing-point of the

The isosceles triangle in front of the picture.

given line is parallel to that side of the isosceles triangle which lies in the given line; *i. e.*, S V is parallel to I *a*; and the side that extends from the station-point to the perspective of the point of distance is parallel to the base of the isosceles triangle; *i. e.*, S D is parallel to *ma*. The triangle in front of the plane of the picture will accordingly also be isosceles, the homologous sides being parallel to the sides of the isosceles triangle behind the picture, and *the perspective of the point of distance will be as far from the perspective of the vanishing-point as the perspective of the vanishing-point is from the station-point; i. e.*, V D is equal to S V.

420. This gives the following easy rule for finding the perspective of a point of distance: —

To find a point of distance.

Given the perspective of a line by its initial-point and the perspective of its vanishing-point, — find the distance from the station-point in front of the picture to the perspective of the vanishing-point, and lay off this distance from the perspective of the vanishing-point along any line that passes through it.

This line will be the horizon of a plane containing the given line, and the point attained will be a point of distance of the given line.

If now a second line be drawn parallel to this horizon, through the initial-point of the given line, it will be the initial-line, or trace, of this plane, and distances laid off upon it from the initial-point may be transferred to the perspective of the given line by drawing lines to the perspective of the point of distance.

421. When the given line is inclined to the Axis, the distance from the station-point to the given vanishing-point is the hypotenuse of a right-angled triangle, of which the station-point is the vertex, the Axis the height, and the distance from the Centre to the given vanishing-point the base. See Fig. 134.

The distance of the vanishing-point from the station-point.

Fig. 134.

When the given line is normal to the plane of the picture, having the perspective of its vanishing-point at the Centre, the distance of this vanishing-point from the station-point is equal to the length of the Axis, and both isosceles triangles are right-angled. See Fig. 135.

Fig. 135.

422. Since the auxiliary plane may be taken in any direction, its horizon may be drawn in any direction through the perspective of the vanishing-point. Any existing line that happens to pass through the perspective of the vanishing-point is accordingly available as such a horizon, a line parallel to it being drawn through the initial-point as an initial-line, or trace. *Conversely*, any existing line that passes through the initial-point may be used as an initial-line, and a horizon may be drawn through the vanishing-point parallel to it, as has already been said (413).

Existing lines available as traces or as horizons.

423. The *locus* of the perspective of the point of distance is then a circle described about the perspective of the vanishing-point as a centre, with a radius equal to the distance of this centre from the station-point. Such a circle will be cut by any given horizon in two points. Each horizon then

The *locus* of the point of distance.

Fig. 136.

contains two such points of distance, one of which makes, with the station-point and the perspective of the vanishing-point, an obtuse-angled isosceles triangle, and the other an acute-angled one. This last is the one commonly employed. See Fig. 136.

When the given line is normal to the picture, the *locus* of the point of distance is a circle described about the Centre, with a radius equal to the Axis. The two points of distance make, with the station-point and the perspective of the point of distance, similar and equal isosceles triangles, right-angled at the Centre, one of which is, in general, as convenient to make use of as the other. See Fig. 135.

The lines S V and S D are optical lines, and make the same angle at the station-point, S, as do the other lines of the systems to which they belong, the same angle as do the given line and the base of the isosceles triangle behind the picture (408).

NOTE.

Surveying. — In surveying, the true angle that any visible horizontal lines make with one another may be determined by finding their vanishing-points on the Horizon, and noting the angle these points subtend at the eye. This may be done either with a theodolite, or by using a Plane Table. See page 273, *c*.

CHAPTER XVIII.

GEOMETRICAL PROBLEMS.

424. Let us now apply the propositions contained in the preceding chapter to the solution of the geometrical problems involved in the chapters that have gone before, adding some problems of a more purely speculative interest, which, in the ordinary practice of perspective drawing, seldom present themselves. We shall find that most of the common problems of Descriptive Geometry, those at least which concern the relations of right lines and planes, are easily solved in Perspective.

425. The following table presents the substance of the principles rehearsed in the previous chapter in a condensed form, as maxims: —

Maxims.

a. A plane passed through the station-point parallel to a plane cuts the plane of the picture in the horizon of the plane (399). Fig. 124. This plane is an Optical plane.

b. A line passed through the station-point parallel to a line pierces the plane of the picture at the vanishing-point of the line (407). Fig. 126. This line is an Optical line.

Hence: —

c. Optical lines and planes, passed through the station-point and through the vanishing-points and horizons of given lines and planes, make the same angles with each other as do the given lines and planes (400, 408). Figs. 120 and 127.

d. If a line is normal to a plane, its vanishing-point lies in a line drawn through the Centre at right angles to the horizon of the plane.

[The theory of projections teaches that if a line is normal to a plane its projection upon a second plane is at right angles to the line in which the two planes intersect.]

e. The initial-line or trace of a plane is parallel to its horizon (398).

f. Parallel planes have the same horizon, and their initial-lines, or traces, are parallel to it and to each other (398). Fig. 119.

g. Parallel lines have the same vanishing-point (406). Fig. 126.

h. Lines lying in or parallel to a plane have their vanishing-points in its horizon (412). Figs. 115 and 129.

i. Lines lying in a plane have their initial-points in its initial-line, or trace (412). Figs. 115 and 129.

Conversely: —

j. The horizon of a plane passes through the vanishing-points of all the lines (or of any two lines) that lie in or are parallel to it (412).

k. The initial-line, or trace, of a plane passes through the initial-points of all the lines (*i. e.*, of any two lines) that lie in it (412).

l. A line parallel to the plane of the picture has its perspective parallel to itself and to the horizons and traces of all planes that pass through or are parallel to it (409). Fig. 121.

Conversely: —

m. All the planes that pass through or are parallel to a line that is parallel to the plane of the picture have their traces and horizons parallel to it and to its perspective (414).

n. The line of intersection of two planes has its initial-point and vanishing-point at the intersection of their traces and horizons (412). Fig. 130.

Conversely: —

o. The traces of all the planes that pass through a line intersect each other at its initial-point (412).

p. The horizons of all the planes that pass through or are parallel to a line intersect each other at its vanishing-point (412).

q. The Centre is the projection of the station-point upon the plane of the picture and the vanishing-point of lines normal to it (393, 411). Fig. 128.

r. The length of the Axis is the distance of the station-point from the centre (393). Fig. 128.

s. The length of the Optical line is the distance of the station-point from the vanishing-point.

Notation.

426. The following system of Notation will be adopted in this chapter. It is conformable to that used in the previous chapters.

The Perspective Plane	P P.
The plane of the picture	*p p*.
The station-point	S.
The station-point when revolved into the plane of the picture	S′, S″, etc.

A line is designated by a Roman capital:—

Horizontal lines, right and left	R, R′, etc.; L, L′, ,,
Lines at 45° (bisecting a right angle)	X, Y; X′, Y′, ,,
Vertical lines (to the zenith)	Z, Z′ ,,
Oblique lines to the right, up and down	M, M′, ,,
Oblique lines to the left, up and down	N, N′, ,,
Lines normal to a given plane (axes of the plane)	T, T′, ,,
Lines normal to the plane of the picture	C, C′, ,,
Oblique lines in normal vertical planes	A, A′, ,,
Lines parallel to the plane of the picture, if horizontal	K, K′, ,,
If inclined down to the right, *dexter*	D, D′, ,,
If inclined down to the left, *sinister*	S, S′, ,,
Sunlight and shadow lines	S.
Other lines	E, J, O, U, etc.
Optical lines	R^O, L^O, N^O, ,,

Finite lines by small capitals	K, Z, C, R, L, ,,
The perspective of a point is designated by an italic letter	*a*, *b*, *c*, *m*, *x* ,,
The vanishing-point of a line by V, with the letter designating the line written above it	V^R, V^L, V^M, ,,
Other vanishing-points	V, V′, V″ ,,

The initial-point of a line by I, with the letter designating the line written above it	I^R, I^L, I^M, etc.
Other initial-points	I, I′, I″, ,,
The points of distance of a line by D, with the letter designating the line written above it	D^R, D^L, D^M, ,,
Other points of distance	D, D′, D″ ,,
A plane is designated by two Roman capitals, indicating its horizontal element and its steepest element	RM, LM, ,,
Horizontal planes	RL, R′L′, ,,
Vertical planes	RZ, LZ, ,,
Normal planes	CK, CD, CS, CZ, ,,
Planes parallel to the picture	KZ, K′Z ,,
Planes of the shadow of lines	SR, SL, SZ, ,,
The horizon of a plane is designated by the letter H, followed by the letters designating the plane	HRN, HLM, ,,
The Horizon	HRL, HCK, or HH.
Horizons of shadows of lines	HSR, HSL, etc.
The initial-line, or Trace, of a plane is designated by the letter T, followed by the letters designating the plane	TRN, TLM, ,,
The Ground-line	TRL, or *gl*.
In the plates a point is shown thus	×
,, ,, ,, a line ,, ,, ,,	
,, ,, ,, the trace of a plane is shown thus	
,, ,, ,, the horizon of a plane is shown thus	
,, ,, ,, a constructive line ,, ,, ,,	

Data.

427. The elements that enter into a perspective problem are, in space, the relative positions of the spectator, the perspective-plane, and the plane of the picture, as given by the position of the Centre and of the Horizon, and the length of the axis of the planes; the size, position, and attitude of the object, as given by the position, magnitude, and directions of the lines and planes to be represented; and, in the plane of the picture, the perspectives of all points and their projection upon that

plane, the perspectives of all finite lines, and the initial and vanishing-points of the infinite lines in which they lie, and the horizons and traces of all planes.

428. When, in any given case, all these things are known, the discussion is complete. The solution of a problem in Perspective consists in showing how, where only some of these elements are given or assumed, others can be determined.

429. Unless the contrary is expressly mentioned, it will be understood that the perspective-plane is vertical, and that the position of the Centre, V^c, that of the Horizon, HH, and that of the Ground-line, $g\ l$, are given, as is also the length of the Axis, $C^o = S\ V^c$, which determines the position of the station-point in front of the picture.

430. The scale is generally supposed to be known; that is to say, the dimension that shall be given, in the drawing, to lines lying in the perspective plane. This depends upon the relative distance of the perspective plane and the plane of the picture from the station-point (94), or upon the scale adopted for the miniature, or model, which is supposed to be substituted for the object itself (396). In that case dimensions can be given by figures. But in purely geometrical discussions they are best given by lines.

431. A single point may be given by its co-ordinates, by its perspective in a given line or plane, or by its perspective and its projection upon a given plane. These all resolve themselves into the case of a point lying in a

given line. If the point is given by its rectangular co-ordinates, or lies in a plane normal to the plane of the picture, it lies in a line normal to the picture; if it lies in an oblique plane, it lies in an oblique line; if it is given by its projection on a given plane, it lies in a line normal to the given plane.

432. Distances are measured from the three planes passing through the Centre, as an origin of rectangular co-ordinates, the three axes of co-ordinates being the Axis, $C^o = SV^c$, normal to the plane of the picture, and passing through the Centre and station-point; the Horizon, $HH = HRL$, normal to the vertical plane which passes through the Centre and the station-point and which is normal to the plane of the picture; and the zenith and nadir line, $ZZ = HCZ$, normal to the horizontal plane, and, like the Horizon, lying in the plane of the picture. These three planes, namely, the Horizontal plane, the Normal plane, and the plane of the picture, are called the *principal* planes.

I. *Problems of Direction.*

These problems are solved by the use of orthographic projections upon the plane of the picture and the principal horizontal plane, according to the common rules of projection. Most of the constructions take place in front of the plane of the picture, which serves as the vertical plane of projection. See Plate XXVI.

In this plate the Horizon, HH, the Zenith line, ZZ, the Centre, V^c, and the length of the Axis, $C^o = SV^c$, are omitted when not needed for the constructions shown.

PROBLEM I. To find the vanishing-point of lines making a given angle with the horizontal-plane, their projection upon that plane making a given angle with the vertical normal plane or with the plane of the picture.

Let the lines M of a given system make the angle β with the horizontal plane, and let their projections upon this plane make the angle α with the normal plane. Then the optical line, M^0, belonging to this system, drawn through the station-point, S, in front of the picture, and having these relations to the principal planes, will pierce the plane of the picture in the common vanishing-point of the lines of the system (**b**).

Given α and β to find V^M.

∴ Let R^0 be the horizontal projection of the line M^0, and let it be drawn horizontally from the station-point, S, at the angle α with the Axis, until it touches the Horizon at the point V^R. This is the vanishing-point of the horizontal projections of the lines M of the given system; that is to say, of the lines parallel to R^0. The required vanishing-point, V^M, will lie vertically above V^R in the plane of the picture, and the triangle, $V^M V^R S$, right-angled at V^R, and having R^0 for its base and M^0 for its hypotenuse, will have the angle at S, between M and R, equal to β (**c**). If now this triangle is revolved about its vertical side, $V^R V^M$, until it coincides with the plane of the picture, S will fall at D. If the line M^0 is drawn in its revolved position, making at D the angle β with the Horizon, the point V^M is easily found at the intersection of M^0, in its revolved position, with the line erected vertically above V^R.

PROBLEM II. If the direction of the given system is

determined by the angle γ which its projection upon the normal plane makes with the Axis, instead of by the angle the line itself makes with the horizontal plane, the vanishing-point can be found by a process analogous to the preceding, as in the plate.

Given the axis of a plane to find its horizon.

PROBLEM III. Given the direction of a plane, or system of planes, by the direction of their axes,— To find the horizon of the system; *that is to say:*—

Given a line by its vanishing-point, V^T,—

To find the horizon of planes normal to it.

∴ Find V^T, the vanishing-point of the axes of the planes (Prob. I. or II.).

The required horizon will be at right angles to a line, $V^T V^C a$, drawn through this vanishing-point and the Centre, V^C; for so much of this line as lies between V^T and V^C is the projection upon the plane of the picture of a line drawn from the station-point to the given vanishing-point, V^T, and the required horizon is the line in which the plane of the picture is intersected by an optical plane drawn through the station-point normal to this line. The line of projection and the line of intersection are then necessarily at right angles to one another (**d**).

It remains only to find the point a.

So much of the line of projection as lies between V^C and a is the projection upon the plane of the picture of a line lying in the optical plane at right angles with the first line. This line passes through the station-point parallel to the given plane, and pierces the plane of the picture at the required point, a, in the required

horizon. These two lines, lying in the air in front of the picture, form at the station-point the vertex of a right-angled triangle, right-angled at S, the hypotenuse of which is the line of projection, $V^T V^C a$.

If this triangle is supposed to be revolved into the plane of the picture around this hypotenuse, the Axis of the picture, $V^C S$, will fall at right angles to it at $V^C S'$. The point a, at the other end of the hypotenuse, is then easily found by setting off at S' the right angle, $V^T S' a$. The required horizon may then be drawn through a, at right angles to $V^T V^C a$.

PROBLEM IV. *Conversely:* Given the horizon of a plane, or system of planes, —

Given a plane to find its axis.

To find V^T, the vanishing-point of lines normal to the system, — that is to say, of their axes.

∴ This simply reverses the operations of Problem III.

If the axis is parallel to the plane of the picture, the horizon of the system of planes lies at right angles to it, and passes through the Centre, V^C. If the system of planes is parallel to the plane of the picture their axes have the Centre, V^C, for a vanishing-point.

PROBLEM V. Given the horizon of a plane and the vanishing-point, V, of a line lying in it, —

To find V', the vanishing-point of a second line, lying in the same plane, making a given angle, ϕ, with the first, or perpendicular to it.

Given V and ϕ to find V'.

Since the optical lines drawn from the two vanishing-points to the station-point must meet at the station-point in the given angle (**m**), and lie

in a plane which cuts the plane of the picture in the given horizon (j), —

∴ Revolve this plane into the plane of the picture, about the given horizon. The projection of the station-point will move from V^C in a line at right angles to the given horizon, and the station-point will fall in its revolved position at S'', the distance, $S''\,a$, being equal to $S'\,a$, determined as in the previous figure, V^CS' being the length of the Axis.

If V is the given vanishing-point, VS'' will be the revolved position of the optical line of the system, and a line making with it the given angle ϕ will be the optical line of the other system and will cut the given horizon at the required vanishing-point, V', as in the figure.

If the angle ϕ is 90°, the two lines will be at right angles.

Given V and V′ to find ϕ.

Problem VI. *Conversely:* Given the direction of two lines, by their vanishing-points, V and V', to find the angle they make with one another in space.

∴ This simply reverses the operations of Problem V.

The given line parallel to the picture.

Problem V. a. If the given line, lying in the given plane, is parallel to the plane of the picture, so that its vanishing-point, V, is at an infinite distance, it will be parallel to the horizon of the plane (1). The line drawn through the station-point parallel to the given line will accordingly also be parallel to the given horizon, and may be drawn parallel to it through S'' in its revolved position, as in the figure. The angle ϕ may then be set off as before.

PROBLEM VI. A. *Conversely:* By a reversal of this process we may ascertain the angle made by a given line, lying in a given plane, with another line lying in the same plane and parallel to the plane of the picture.

PROBLEM VII. Given the direction of two planes by their horizons to find the angle between them.

To find the angle between two planes.

[The angle made by two planes is the supplement of the angle made by their axes.]

∴ Find the vanishing-points of the axes of the two planes (Prob. IV.), and then find the angle made by these axes. (Prob. VI.)

When the planes are normal.

PROBLEM VII. A. If the planes are normal to the plane of the picture, they make the same angle with each other as do their horizons.

When their horizons are parallel.

PROBLEM VII. B. If the elements in the two planes which are parallel to the plane of the picture are parallel to one another, *i. e.*, if the horizons of the planes are parallel, as in Fig. 123, Plate XXV., the problem may be solved by finding the angle between the planes themselves instead of that between their axes, as follows:—

Find the points a and a', in the horizons of the two planes, as in Prob. III. The angle aSa', revolved into the plane of the picture at $aS'a'$, will obviously be the required angle.

PROBLEM VIII. Given the direction of a line by its vanishing-point, and that of a plane by its horizon, to find the angle between them.

To find the angle between a line and a plane.

[This angle is the complement of that made by the line and the axis of the plane.]

∴ Find the vanishing-points of the axis of the given plane (Prob. III.), and then find the angle between this axis and the given line (Prob. VII.).

∴ Another way: Project the given line upon the given plane (Prob. XXIX.), and then find the angle between the line and its projection (Prob. VI.).

PROBLEM IX. Given the direction of a line, or system of lines, M, by the vanishing-point, V^M, —

To find (1) the optical line M^O, *i. e.*, the distance from the station-point, S, to the vanishing-point, V^M, (2) a point of distance, D^M, and (3) the *locus* of the points of distance of the system.

To find a point of distance and its *locus*.

(1.) The optical line, from the station-point to the vanishing-point, is the hypotenuse of a triangle, right-angled at the centre of the picture, V^C, of which the Axis, V^CS, is one side. By revolving this into the plane of the picture around its base, V^CV^M, the distance, $SV^M=M^O$, is easily found.

(2.) This distance laid off, right or left, upon the horizon of any plane which contains M, and which consequently passes through V^M (**g**), gives the points, D^M, which are called points of distance of M. Their distance from V^M shows the distance of the station-point from V^M.

(3.) As the horizons of all planes containing M pass through V^M (**j**), all possible points of distance of the system M will lie in a circle, of which V^M is the centre, and the distance, $V^MS=M^O$, the radius. This circle is the *locus* of D^M.

∴ If the given lines belong to the system C, normal to the plane of the picture, their vanishing-point is the

Centre, V^C, and the radius of the circle, which is the *locus* of D^C, is the Axis, V^CS. The point of distance, D^C, accordingly, always shows at once how far the station-point is from the picture. The length of the Axis is equal to the distance, $V^CD^C = C^O$.

The operations by which the remaining problems are solved are conducted in the plane of the picture by means of vanishing-points, initial-points, horizons, traces, and points of distance which have been previously determined.

II. *Problems of Dimension and Position.*

PROBLEM X. To cut off a given length, M, from a line given in perspective by its initial-point, I^M, and its vanishing-point, V^M, —

∴ Draw through the initial-point a line, at random, to represent the trace of a plane passing through the line, and draw a parallel line through the vanishing-point to represent the horizon of this plane (**a**, **j**, **e**). Find the *locus* of the points of distance of the given line (Prob. IX.). The points in which it cuts the assumed horizon will be points of distance of the given line.

To measure off lengths on a perspective line.

Lay off upon the initial-line, or trace, at *m*, the length to be cut off upon the line given in perspective, and from the point thus ascertained draw a line *across* the given line to the opposite point of distance. The length cut off upon the given line will be the required perspective length.

For, in the figure, the line $V^M D^M$ is, by construction, parallel to $I^M m$, SV^M (in the air in front of the picture) to $I^M a$ (**b**), and SD^M (also in the air), to $m\,a$ (**b**). And since the triangle, $SD^M V^M$ (in front of the picture) is isosceles, $V^M D^M$ having been taken equal to SV^M, the triangle shown in perspective at $I^M m\,a$ is isosceles also, the corresponding sides of the two triangles being parallel, and $I^M a$ is equal to $I^M m$.

It is to be noticed that these triangles are reversed in position.

PROBLEM XI. *Conversely:* Given a finite line M in perspective, or two points, with the initial-point and vanishing-point of the infinite line in which they lie,—

To find the true length of a given perspective line.

To find the length of the line, or the distance apart of the two points.

Let a and a' be the given points, and I^M and V^M the given initial-point and vanishing-point.

∴ Through I^M and V^M draw parallel lines in any convenient direction. These will represent the trace and horizon of a plane containing the given points (**e, j, k**) (422). Find a point of distance, D^M (Prob. IX.).

From D^M draw lines through a and a', until they intersect the trace at m and m'.

The distance $m\,m'$ is the true distance of the points $a\,a'$, or the length M of the line $a\,a'$.

If in this and the preceding problem the given points or line lie in a line normal to the plane of the picture, then I^C, V^C, and D^C take the place of I^M, V^M, and D^M.

PROBLEM X. A. To cut off given lengths from lines parallel to the plane of the picture.

The same, when the line is parallel to the picture.

It is necessary, in order to fix the position of the given line, that the initial-point and vanishing-point of an auxiliary line intersecting the given line should also be given.

∴ Let p be a point in the given line, $p\,a$, and I and V the initial-point and vanishing-point of an auxiliary line intersecting it at this point. Draw parallel lines through V and I to represent the horizon and trace of a plane containing both lines. They will be parallel to the given line (**1**), and any distances, Im, $m\,m'$, taken upon the initial-line, may be intercepted upon the given line at $p\,a$ and $a\,a'$ by lines parallel in space (**c**), drawn from I, m and m', to V, as in the figure.

PROBLEM XI. A. *Conversely:* Given a finite line parallel to the plane of the picture, with the initial-point and vanishing-point of an auxiliary line passing through it, —

To find the real length of the given line.

∴ This problem is simply the reverse of the preceding.

PROBLEM XII. To find the perspective of a point given by its co-ordinates.

To find the perspective of a point.

∴ Let the origin of co-ordinates be the Centre, V^c, and let the horizontal, vertical, and normal co-ordinates of the point be K, Z, and C, respectively.

The normal co-ordinate will lie in an infinitely long line, C, passing through the required point and piercing the plane of the picture at its initial-point, I^C. Hence:—

Lay off in the plane of the picture, horizontally and vertically, from the origin, V^C, the co-ordinates K and Z. They give the position of I^C, the initial-point of the normal line passing through the required point. Cut off upon this line the distance, C (Prob. X.). The point obtained will be the perspective of the required point, *a*, as in the figure at A. The point I^C is the orthographic projection of the point *a* upon the plane of the picture.

[The orthographic projection of a point upon the plane of the picture, the perspective of the point, and the centre of the picture, lie, of course, in the same straight line.]

The horizon and trace of the auxiliary plane may be taken at random, as at A, or, as is generally more convenient, and as is shown at B, may be taken as coinciding in direction respectively with K and the Horizon, HH, or with Z, and the zenith line, ZZ, as shown in the figure at C.

To find the co-ordinates of a given point.

PROBLEM XIII. *Conversely:* Given a point by its perspective, *a*, and its projection on the plane of the picture, I^C, to find its co-ordinates.

The horizontal and vertical distance of I^C from V^C, in the plane of the picture, are the co-ordinates K and Z, and the real length of the line $I^C a$ (Problem XI.) is the third co-ordinate, C.

III. *Problems of Planes, Lines and Points.*

PROBLEM XIV. Given a plane by its horizon and trace, —

To draw a line in it, either at random, or having a given direction parallel or inclined to the plane of the picture, or parallel to a given line lying in or parallel to the given plane, or making a given angle with such a line, and to find the vanishing-point and the initial-point of the line so drawn.

To draw a line in a plane.

In all these cases the vanishing-point, which necessarily lies in the horizon of the given plane (**h**), either may be assumed, or is given, or may be found by Problem V.

The initial-point necessarily lies in the given initial-line or trace (**k**).

If the line is required to pass through a given point, *a*, —

∴ Draw the perspective of the required line through the perspective of the given point and the vanishing-point. The initial-point is at the intersection of this perspective line with the given initial-line (**i**).

Through a given point.

If the line is required to pass through two points in the plane, or a finite line given by its terminal points, —

∴ Pass the perspective of the line through the perspective of the given points; its initial-point and vanishing-point will lie at its intersection with the trace and horizon of the given plane, respectively, as at I′ and V′.

Through two points.

If the line is required to be parallel to the plane of the picture, its perspective will lie parallel to the horizon of the given plane (**1**), and its initial-point and vanishing-point will be at an infinite distance.

Parallel to the picture.

PROBLEM XV. *Conversely:* Given a line by two points, or by its initial-point and vanishing-point, —

To pass a plane through it at random, or parallel to a given line, or normal to a given plane, or to the plane of the picture.

To pass a plane through a line.

If the line is given by two points, they are given, virtually, as lying in parallel normal lines (Prob. XII.). The line, then, lies in a normal plane, and its initial-point and vanishing-point are easily found (Prob. XIV.) (**h**, **i**).

If the required plane is to be normal to a plane it will be parallel to the axis of the plane; if normal to the plane of the picture it will be parallel to the axis of the picture. These conditions are comprised, then, in the condition that the required plane shall contain a line given by its initial-point and its vanishing-point, and shall be parallel to a second line, the vanishing-point of which is either given or easily ascertained (Prob. IV.). Hence: —

Parallel to a given line.

∴ Draw a line through V, the vanishing-point of the given line, and V′, the vanishing-point of the line to which the required plane is to be parallel, for the horizon of the required plane, and draw a second line, parallel to this horizon, through the initial-point of the given line, for the required initial-line or trace.

If it is normal to the plane of the picture the horizon will pass through the centre, V^C. Normal to the picture.

If the required plane is to be drawn through the given line at random, any two parallel lines drawn through its initial-point and vanishing-point will represent, respectively, the trace and horizon of a plane passing through it, as in Problems X. and XI. (413). At random.

PROBLEM XVI. To pass a line through a point given by its perspective, and lying in a plane given by its horizon and trace, that shall lie in the given plane and shall be parallel to a second plane. To draw in a plane a line that is parallel to a second plane.

The required line will lie in one plane and be parallel to the other. Accordingly its vanishing-point will lie in both their horizons (**h**); that is to say, at the intersection of their horizons. Hence: —

∴ Draw a line from the intersection of the horizons of the two planes, as a vanishing-point, through the perspective of the given point to the initial-line of the plane in which it lies. The point in which it meets this line will be the initial-point of the required line.

PROBLEM XVI. A. If the second plane is parallel to the first, the line can of course be drawn in any direction.

PROBLEM XVI. B. If the second plane is parallel to the plane of the picture the required line will also be parallel to the plane of the picture, and will be drawn parallel to the horizon of the plane in which it lies (**1**).

PROBLEM XVI. C. If the two planes, though not parallel, have those elements parallel which are parallel to the plane of the picture, so that the horizons of the two planes are parallel, as in Problem VII. B, and in Fig. 123, Plate XXV., the required line will still be drawn parallel to the horizon of the plane in which it lies. It will then still be directed towards the infinitely distant intersection of the two parallel horizons **(1)**.

To pass a line through a point in space.

PROBLEM XVII. Given a point by its co-ordinates, or by its projection, or by its perspective and a line or a plane in which it lies, —

To pass a line through it at random, or having a given direction parallel or inclined to the plane of the picture, or parallel to a given line, or normal to or making a given angle with a given line in a given plane, or normal to a given plane, or to the plane of the picture.

All of these different ways of determining a point in position have been reduced by the previous problems to the case of a point given by its perspective and by the initial-point and vanishing-point of a line passing through it. So, also, all the conditions attached to the required line have been reduced to the conditions that its vanishing-point is known; that is to say, it is either assumed, is given, or may be found.

The problem is then simply this: —

Given a point by its perspective, *a*, and the initial-point, I, and vanishing-point, V, of a line passing through it, —

To pass a line through it in the direction given by a second vanishing-point, V′, and to find I′, the initial-point of the required line.

In a given direction.

∴ Pass a plane through the given line parallel to the required line (Prob. XV.). The given point will lie in this plane.

In this plane draw a line through the given point having the required direction (Prob. XIV.).

Problem XVIII. To pass a line through two points given by their co-ordinates, or otherwise, and to find its initial-point and its vanishing-point.

To pass a line through two points.

Each point lies, or may be found to lie, in a given line normal to the plane of the picture. The two given points lie, then, in a given normal plane, and the initial-point and vanishing-point of a line joining them will lie in the trace and horizon of that plane (Prob. XIV.).

Problem XIX. Given three points by their co-ordinates, or otherwise, —

To pass a plane through three points.

To pass a plane through them:

∴ Pass a line through the first and second, and another through the second and third (Prob. XVIII.).

Pass the required plane through the first line parallel to the second (Prob. XV.).

Problem XX. Given two lines, in the same plane, to pass a plane through them, and find their point of intersection.

To pass a plane through two lines.

∴ Draw a line through V and V′, the

vanishing-points of the two lines, for the horizon of the plane, and through their initial-points, I and I′, for its trace. The intersection of their perspectives will be the perspective of their intersection.

[The condition that the lines lie in the same plane necessarily implies that they intersect one another, or will do so if prolonged, and that the line joining their initial-points is parallel to the line joining their vanishing-points.]

To determine whether two lines intersect.

PROBLEM XXI. *Conversely:* to discover whether two lines given in perspective intersect in space; that is to say, whether they lie in the same plane:—

∴ If the line joining their initial-points is parallel to the line joining their vanishing-points, they do; otherwise, not.

To pass a plane through two parallel lines.

PROBLEM XX. A. If the lines are parallel to one another, the line joining their initial-points will be the trace of the required plane, and its horizon will be drawn parallel to this trace through their common vanishing-point. There will, of course, be no point of intersection.

To pass a plane through two lines when one is parallel to the picture.

B. If one of the lines is parallel to the plane of the picture it will be parallel to the horizon and to the trace of any plane that passes through it (1). The required horizon and trace will accordingly be drawn parallel to this line through the initial-point and vanishing-point of the other line.

C. If both lines are parallel to the plane of the picture, an auxiliary line must be drawn from a point

in the first line to a point in the second, and its initial-point and vanishing-point found (Prob. XVIII.). The trace and horizon of the required plane may then be drawn through the initial-point and vanishing-point of this auxiliary line, parallel to the given lines.

The same when both lines are parallel to the picture.

PROBLEM XXI. A. *Conversely:* to discover whether two lines given in perspective intersect with one another, one being an oblique line, given by its initial-point, I, and its vanishing-point, V, and the other a line parallel to the plane of the picture, given in position by the projection upon the plane of the picture at I^c, of some point in it, a, the centre, V^c, being also given.

To determine whether two lines intersect when one is parallel to the picture.

∴ Draw through a, the perspective of the given point, its line of projection from V^c to I^c, and through the same point an auxiliary line, parallel to the given oblique line. These two lines will lie in a normal plane whose horizon will pass through V and V^c. Its trace, drawn through I^c parallel to this horizon, will determine I', the initial-point of the auxiliary line. If a second plane is now passed through the auxiliary line and the given line which is parallel to the picture, its trace T may be drawn through I' parallel to the given parallel line. The auxiliary line and the parallel line will both lie in this second plane.

Now if the given oblique line really intersects the given parallel line at b, it also will lie in this plane, and the trace T will pass through its initial-point, I, as shown at A.

If, then, the given lines intersect one another, the

two initial-points I and I′ will be equidistant from the given parallel line.

But if the given lines do not intersect at *b*, then, as in XXI. B, the oblique line will not lie in the same plane with the parallel line and the auxiliary line, and I will not lie in the trace T.

In the figure the oblique line is plainly on the hither side of the parallel line.

PROBLEM XXII. Given two planes, by their traces and horizons, —

To find the intersection of two planes.

To find their line of intersection.

∴ Draw a line from the intersection of the two traces, as its initial-point, to the intersection of the two horizons, as its vanishing-point (**n**).

The same, when their horizons are parallel.

PROBLEM XXII. A. If the two planes, as in Problem VII. B, and in Problem XVI. C, have those elements parallel which are parallel to the plane of the picture, so that their horizons and traces are all parallel, like those shown in Figs. 119 and 123, Plate XXV., their line of intersection will be parallel to their traces and horizons, being directed to their infinitely-distant points of intersection.

To find the position of this line an auxiliary plane must be passed across the two given planes, and its lines of intersection found. The point in which these lines of intersection meet will be one point in the intersection of the given planes, and the required line of intersection may be drawn through the point parallel to the given traces and horizons.

Such an auxiliary plane will have its horizon, of

course, parallel to its trace; and as such a plane may take any direction, any two parallel lines may be held to represent its horizon and trace. Hence:—

To find the required line of intersection of two planes whose traces and horizons are all parallel,—

∴ Draw any line across the given traces as the trace of the auxiliary plane, and any other line parallel to the first across the given horizons as its horizon. Draw the two lines in which the auxiliary plane intersects the given planes (Prob. XXII.), and through the point where they cross one another draw the required line parallel to the given traces and horizons.

PROBLEM XXIII. Given three planes,—

To find their point of intersection.

To find the point of intersection of three planes.

∴ Find the line of intersection of the first and second, and the line of intersection of the second and third (Prob. XXII.). Then find the point of intersection of these two lines of intersection (Prob. XX.).

PROBLEM XXIV. Given a line and a plane,—

To find the point in which the line pierces the plane.

To find where a line pierces a plane.

∴ Pass through the given line, an auxiliary plane, at random (Prob. XV.), and find its line of intersection with the given plane (Prob. XXII.). This line of intersection, and the given line, will both lie in the auxiliary plane. Find their point of meeting (Prob. XX.). This point will lie in the given line, and also in the given plane, and will be the point of puncture required. The auxiliary plane has

been taken as vertical, this being, in the majority of cases, the most convenient position in which to assume such a plane.

The same, when the line is parallel to the picture.

PROBLEM XXIV. A. If the given line is parallel to the plane of the picture, its position is fixed by that of some point in it, and a normal plane may be passed through it, as in Problem XV.

IV. *Problems of Projection, etc.*

To find the projection of a point upon a plane.

PROBLEM XXV. Given a point by its perspective, a, and the vanishing-point and initial-point of the normal or oblique line in which it lies, —

To find its orthographic projection upon a given plane.

∴ Find the vanishing-point of lines normal to the given plane (Prob. IV.), and draw through the given point such a line (Prob. XVII.).

Find the point in which this normal line pierces the given plane (Prob. XXIV.).

This point is the required projection. The normal line is the line of projection.

The same, when the plane is parallel to the picture.

PROBLEM XXV. A. If the given plane is parallel to the perspective plane it will be given in position by the perspective of some point in it with its projection, I^p, upon the plane of the picture. The centre, V^c, will be the vanishing-point of the line of projection (**q**).

∴ Draw the line of projection of the point in the given plane, and also the line of projection of the point in the given line, and find their initial-points (Prob.

XVII.). The two lines of projection will lie in a normal plane, the trace of which will pass through their two initial-points. This normal plane will intersect the given plane in a line parallel to this trace, and the point in which this line cuts the line of projection of the point in the given line, is the point required.

PROBLEM XXVI. Given a line in perspective, to find its projection upon a given plane.

To find the projection of a line upon a plane.

∴ Find the projection upon the given plane of some point of the line (XXV.).

Pass through the given line an auxiliary plane normal to the given plane (Prob. XV.). Find the line in which this auxiliary plane intersects the given plane (Prob. XX.). This line is the required line of projection.

PROBLEM XXVII. Given a point and a line in perspective,—

To find the distance between a point and a line.

To find upon the given line its point of nearest approach to the given point, and the distance between them.

∴ Pass through the given point a plane normal to the given line (Prob. III.), and find the point in which the given line pierces this plane (Prob. XXIV.). This point is obviously the required point, and its distance from the given point (Prob. XI.), the required distance.

PROBLEM XXVIII. Given two lines in perspective, to find their points of nearest approach, and the distance and direction from one another of those points.

To find the distance between two lines.

∴ Pass through one line, I′ V′, an auxiliary plane parallel to the other, I V, (Prob. XV.), and find the projection of the second line upon this plane, to which it is parallel (Prob. XXVI.). This projection and the first line will both lie in this plane.

Find the point *a* in which the projection of the second line intersects the first line (Prob. XX.). This is one of the required points, and the point *b* of the second line thus projected is the other.

Find the length of the line *a b* joining these two points (Prob. XI.); this is the required distance of the two points, and the direction of the line joining them, which is the direction of lines normal to the auxiliary plane, is the required direction.

PROBLEM XXIX. Given the perspective of a finite line, and its direction by the vanishing-point of the line in which it lies:—

To divide a line in any given ratio. To divide it in any desired ratio, or into equal parts, or proportionally to another line.

[This problem is analogous to the problem of cutting a given length from a perspective line (Prob. X.). But in that case the given line is indefinite in length, and the auxiliary triangle is an isosceles triangle, of which the adjacent sides are divided into equal parts by lines drawn parallel to the base. In the present case the perspective line is of definite length, and the auxiliary triangle is scalene, the adjacent sides being divided proportionally.]

∴ Through the given vanishing-point, draw a line in

any convenient direction as the horizon of a plane passing through the given line (j).

Through either end of the given finite line draw a line parallel to the horizon thus assumed, and lay off upon this line, as a line of measures, the given proportional or equal parts at any convenient scale.

Draw through the extreme point thus ascertained a line passing through the other end of the given finite line, and prolong it until it intersects the assumed horizon. Towards this point of intersection, as a vanishing-point, draw lines from the points taken upon the line of measures. The points at which they cut the given line will be the required points of division.

The same, when the line is parallel to the picture.

Problem XXIX. a. If the given line is parallel to the plane of the picture, its perspective will be proportional in all its parts to the given line, and it may be divided, as the line itself would be, according to the methods of plane geometry.

SUMMARY.

These twenty-nine Geometrical Problems may more briefly be stated as follows: —

Besides the *data* here mentioned, the position of the centre, V^c, and the length of the Axis, C^o, which fixes the position of the station-point, S, are understood to be given, though these are not needed for the solution of the problems numbered from XII. to XXIV., nor for Problem XXIX.

Problems of Direction.

Problem.	Given.	Required.
I.	a, β.	V.
II.	a, γ.	V.
III.	V^T.	H.
IV.	H.	V^T.
V.	H, V, ϕ.	V′.
V. A.	*The same*, when V is at infinity.	
VI.	V, V′.	ϕ.
VI. A.	*The same*, when V is at infinity.	
VII.	H, H′.	ϕ.
VII. A.	*The same*, when the planes are normal.	
VII. B.	*The same*, when H and H′ are parallel.	
VIII.	H, V.	ϕ.
IX.	V.	The length of the optical line, and *locus* of D.

Problems of Dimension and Position.

Problem.	Given.	Required.
X.	I^M, V^M; m.	M.
X. A.	*The same*, when the given line is parallel to $p\,p$, I and V being at an infinite distance.	
XI.	I^M, V^M; M.	m.
XI. A.	*The same*, when M is parallel to $p\,p$, I and V being at an infinite distance.	
XII.	K, C, Z; V^C.	a.
XIII.	a, I^C, V^C.	K, C, Z.

Problems of Lines, Planes, and Points.

Problem.	Given.	Required.
XIV.	H, T; a.	V, I.
XIV. A.	H, T; a, a'.	V, I.
XV.	I, V; or a, a'.	H, T.
XV. A.	*The same*, when the required plane is to be parallel or normal to the given line, or parallel to $p\,p$.	
XVI.	H, T; a, H′.	I, V.
XVI. A.	*The same*, when H′ is parallel to H.	
XVI. B.	*The same*, when I, V, and H′ are at an infinite distance.	
XVII.	a, I; V, V′.	I′.
XVIII.	a, a'.	I, V.
XIX.	a, a', a''.	T, H.
XX.	T, H; I, V; I′, V′.	The point of intersection of the two lines.

PROBLEM.	GIVEN.	REQUIRED.
XX. A.	*The same*, when $V=V'$; when V is at an infinite distance; when V and V' are both at an infinite distance; *i. e.*, when we have one line or both parallel to $p\,p$.	
XXI.	I, V; I', V'.	Do the lines intersect?
XXI. A.	*The same*, when we have one line or both parallel to $p\,p$.	
XXII.	T, H; T', H'.	The line of intersection of the two planes.
XXII. A.	*The same*, when H and H' are parallel.	
XXIII.	T, H; T', H'; T'', H''.	The point of intersection.
XXIV.	T, H; I, V.	The point of intersection.
XXIV. A.	*The same*, when I and V are at an infinite distance; *i. e.*, when the line is parallel to $p\,p$.	

Problems of Projection.

PROBLEM.	GIVEN.	REQUIRED.
XXV.	I, V; a; T, H.	The projection of the point on the plane.
XXV. A.	*The same*, when T and H are at an infinite distance; *i. e.*, when the plane is parallel to $p\,p$.	
XXVI.	I, V; T, H.	The projection of the line on the plane.
XXVII.	I, V; a.	The distance of the point from the line.
XXVIII.	I, V; I', V'.	The distance apart of the two lines.
XXIX.	V^{M}, M.	To divide M in a given ratio.
XXIX. A.	*The same*, when the line is parallel to $p\,p$.	

CHAPTER XIX.

THE PRACTICAL PROBLEM.

433. After all, the question remains, How is one to go to work, in a given case, to make a perspective drawing. The shape and size of the object to be drawn, a building for instance, are, of course, supposed to be given, with the scale to be employed in the plane of the picture. The scale either may be assumed or may be determined by comparing the relative distances of the object and of the plane of the picture from the spectator (94).

The data.

434. The next thing to be determined is the *attitude* of the object; that is to say, the angle its principal lines shall make with a line drawn from the eye to the object. The direction of this line is in general purely arbitrary, being so chosen as to exhibit the building or other object in its best aspect. The plane of the picture is generally taken at right angles to this line, which then becomes the Axis of the picture, some point near the middle of the object being then at the Centre. But when it is possible, by giving this Axis a slightly different direction, to bring

The attitude of the object.

the principal lines of the object at 45° with the plane of the picture, making the Centre, V^C, coincide with V^X, the "vanishing-point of 45°," it is best to do so. This adjustment is exemplified in Fig. 14, Plate V., and in Fig. 138, Plate XXVII.

Best at 45°,

435. When one side of the object is nearly parallel with the picture, it is often desirable, and sometimes necessary, as has been explained in paragraph 276, to make it exactly parallel.

or parallel.

436. It is generally desirable to have the plane of the picture as far from the eye as possible, which is equivalent, for any given scale, to having the object itself as distant as possible, the distance of the picture multiplied by the denominator of the fraction expressing the scale, giving the distance of the object (94). If, for instance, the scale employed in the drawing is that of one eighth of an inch to the foot, that is to say, one ninety-sixth full size, the object, or that portion of it that lies in the plane of projection, or plane of measures, will be ninety-six times as far away as the drawing.

The Axis to be as long as possible.

The further the station-point is from the picture the easier will it be for the spectator to occupy it, and the less will be the apparent distortion of the drawing if he fails to occupy it exactly.

437. But setting the station-point far from the picture is equivalent to setting the vanishing-points far apart, so that practically the first thing to do after the attitude of the object is

i. e., V^L and V^R to be as far apart as possible.

chosen, and the angle it is to make with the plane of the picture determined, is to fix the distance apart of the principal right-hand and left-hand vanishing-points, V^R and V^L. These points, which of course lie in the Horizon, are generally set at the extreme limits of the table or drawing-board upon which the work is to be done. See Fig. 137, A, Plate XXVII., in which the assumed attitude of the object is shown by two lines drawn at right angles to one another, making the given angles with the picture. These lines may be drawn in any convenient place, it makes no difference where.

438. As the principal horizontal lines, R and L, vanishing at V^R and V^L, upon the Horizon, are generally at right angles, the station-point S is, in plan, generally at the vertex of a right-angled triangle, of which the line $V^L V^R$, in the plane of the picture, is the hypotenuse. The *locus* of S is accordingly a horizontal semicircle, of which the line $V^L V^R$ is the diameter. The next step after fixing these points is, then, to describe such a semicircle, and to find upon it the point S, such that the line SV^R will be parallel to the right-hand side of the object, and SV^L to its left-hand side. A perpendicular dropped from this point upon the line $V^L V^R$ will give the position of the Centre, V^C; a diagonal line bisecting the right angle will give the point V^X, the "vanishing-point of 45°" (44); and the lines SV^R and SV^L revolved into the plane of the picture will give respectively the right and left-hand points of distance, D^R and D^L.

Plate XXVII. Fig. 137.

S, V^C, V^X, D^L, D^R, and D^X.

D^X, the point of distance of the diagonal line, may be obtained at the same time, if desired, by revolving SV^X into the plane of the picture, as in Plate IV., Fig. 11.

439. If shadows are to be cast, and the vanishing-point of shadows, V^S, lies beyond either of the principal vanishing-points, as it does in the figure, room must be allowed for this vanishing-point also. This space, however, may be saved by taking the sun in the plane of the picture, as in Fig. 36, Plate VIII., with the vanishing-point of shadows at an infinite distance (184).

440. If the object is set just at 45°, as has been recommended, and as is done in Plate V., and in Fig. 138, Plate XXVII., its two sides making equal angles with the plane of the picture, it is not necessary to describe the semicircle at all. The Centre will be half way between V^L and V^R, the station-point will be the same distance in front of the Centre, V^X will coincide with V^C, and D^L and D^R may be found as before. They will be almost exactly two fifths of the distance from the Centre to either vanishing-point, as shown.

The object at 45°. Fig. 138.

441. All these operations are conducted in plan, the paper at 137, A, and 138, A, representing the ground plane, or horizontal plane of projection, the line $V^L V^R$ being the projection of the Horizon. This line also represents both the plane of the picture, *pp*, seen edgewise, and the ground line, *g l*, in which the plane of the picture cuts the ground plane.

The ground plane.

442. If now the plane of the picture is revolved into

the plane of the paper about the horizon, the points V^R, V^L, V^X, D^R, D^L, and V^C will retain their positions, the ground line, *g l*, will appear in the plane of the paper at some distance below the Horizon, and parallel to it, and the station-point, S, will be in the air in front of the picture, opposite the Centre, as shown in Plate XXV., and in Fig. 13, Plate IV.

The plane of the picture.

443. Vertical lines, erected in the plane of the picture at V^L and V^R, will now establish HRZ and HLZ, the horizons of the principal vertical planes, and the vanishing-points of the inclined lines M, M′, N, and N′, lying in or parallel to these planes, may be fixed by drawing lines at D^L and D^R, that make, with the Horizon, the same angles, β and β', that the lines themselves make with the horizontal plane. The points in which these lines intersect the horizons of the vertical planes will give the vanishing-points V^M, $V^{M'}$, V^N, and $V^{N'}$ (88). If, as is generally the case, the roofs have the same slope, β and β' will be equal.

V^M, $V^{M'}$, V^N, and $V^{N'}$.

444. These points being determined, the horizons of inclined planes, HRN, HRN′, HLM, and HLM′, can be drawn whenever they are needed, and V^P and $V^{P'}$, the "vanishing-points of hips and valleys," can be ascertained, as in Plates I., II., III., IV., and V.

HRN, HLM, etc., V^P and $V^{P'}$.

445. In the same way V^S, the vanishing-point of sunlight or of shadows may be fixed by determining first a point, $V^{R'}$ (or $V^{L'}$), the vanishing-point of the horizontal projection of the rays of light, as in Fig. 137, A. $D^{R'}$ (or $D^{L'}$) can then be obtained by revolving the horizontal line $SV^{R'}$ (or SV^L)′ into

V^S.

the plane of the picture. The position of V^S, immediately above or below $V^{R'}$ (or $V^{L'}$), can then be fixed by laying off at $D^{R'}$ (or $D^{L'}$) the real angle made by rays of light with the ground-plane, and the horizons of the planes of invisible shadows, and the vanishing-points of the lines of visible shadow obtained by drawing lines from V^S to the various vanishing-points, and noting their intersection with the horizons previously determined, as is done in Figs. 34, 35, 36, and 38, Plate VIII. (181).

446. The station-point, S, and the principal vanishing-points, V^R and V^L, being once established, all these other horizons, vanishing-points, and points of distance may be determined either at once, as in Fig. 38, before the drawing is begun, or from time to time, as they are needed, during the progress of the work. In either case they are the necessary scaffolding, so to speak, without which the various constructions cannot be carried on.

Preliminary operations.

447. If the drawing is to be made in Parallel Perspective the extreme vanishing-points become the vanishing-points of diagonals, which will coincide with the points of distance, as in Plate VI. It will generally be convenient to establish at once points of half or quarter distance, as shown in Fig. 21, in that Plate (142), and in Fig. 94, Plate XX. (333).

Parallel perspective.

448. All this preliminary work is concerned solely with the *direction* of lines and planes, not with their position. Before constructing a perspec-

The position of the object, horizontally.

tive drawing by their aid it is necessary to determine also the *position* to be assigned to the object; that is to say, to some prominent point in it. The point generally selected is the lower end of the nearest corner.

The first thing to be done is to determine how far to the right or left of the Centre this point shall be set. It is generally on the right if the left-hand side of the object is to be made prominent, and *vice versa*. In Fig. 137 the position of the front corner is assumed, and, the building being rather a large one, it is set considerably to the right of the Centre. In Fig. 138 both the attitude and the position of the building are determined upon the orthographic plan at A.

449. As everything in a perspective drawing is more or less distorted, according as it is more or less removed from the Centre, opposite the eye (260), it is of the first importance that the object shall be as near the Centre as possible. The Centre does not necessarily, as has been said, coincide with the middle of the picture. But it is desirable to have it do so, as nearly as is convenient; and, as the object itself naturally occupies the middle of the picture, if the Centre is in the middle of the object this point also is generally attained.

The Centre.

450. But it is often worth while to throw the object considerably to one side of the Centre; that is to say, to have the Centre quite out of the middle of the picture, if by so doing the practical advantages and convenience of having the principal right and left-hand vanishing-points at 45° can be secured.

In Fig. 15, for instance, Plate V., the Centre is near the left-hand corner of the little house. But the main lines of the building do not materially differ from those of Fig. 12, Plate IV., in which the Centre is near the right-hand corner.

451. Where, as in Fig. 15, two objects are to be shown, it is of course impossible that both should occupy the same place at the Centre. But it is generally practicable to have one of them, or both of them, if, as in this case, they are parallel to one another, at 45° with the picture.

Two objects.

452. The position to be given to the object, horizontally, having been determined, the next thing is to draw a Perspective Plan of it; *i. e.*, to put into perspective its horizontal projection.

The perspective plan.

453. The horizontal plane upon which the perspective plan of the object to be represented is supposed to be drawn, is called the Ground-Plane, and its initial line, *i. e.*, the trace in which the ground-plane cuts the plane of the picture, is called the ground-line, or line of horizontal measures, as has been said. It is convenient, for many reasons, to have this as far as may be below the Horizon (46, 101), and it is well to draw it upon a separate piece of paper, covering the lower part of that upon which the drawing is to be made, so that the construction lines that lie in its neighborhood may not deface the picture, and so that they may be removed and used again, if necessary, instead of being erased.

The ground-line, *g l*.

The sunk-perspective plan.

454. This is shown in Figs. 137, B, and 138, B, in which the Horizon, with the various vanishing-points and points of distance, are transferred directly from Figs. 137, A, and 138, A, and the ground-line, *g l*, drawn in an inch or two lower down.

In practice the figures A and B would be drawn one over the other, on the same paper. But in the plate the constructions by which the position of the vanishing-points and points of distance is determined, both upon the horizontal plan and in the plane of the picture, are drawn at A, and the perspective itself, and the perspective of the plan, both of which are in the plane of the picture, are drawn at B. This avoids confusion, and allows all the construction lines to be shown in full, which in practice would be drawn only in part.

The starting-point.

455. It is customary to have the front corner of the building, or other object to be drawn, lie in the plane of projection, or, which comes to the same thing, to have the imaginary model touch the plane of the picture (396) as in these figures. In the perspective plan, then, the horizontal projection of this corner will lie in the ground-line, as shown at the point I. Lines drawn from this point, as an initial-point, to the principal horizontal vanishing-points, V^R and V^L, are the front lines of a perspective plan. They are infinite lines, upon which the horizontal dimensions of the object can be cut off by means of the points of distance already established, the ground-line serving as a line of horizontal measures.

456. The length of the right-hand side of the building, or other object, with its subdivisions, being then laid off upon the ground-line to the right of this point, and of the left-hand side towards the left, may be transferred to these infinite perspective lines by drawing lines *across* them to the right-hand and left-hand points of distance respectively.

Horizontal lines inclined to the picture.

457. In drawing the perspective plan constant use may be made of the diagonal line bisecting the right-angle, and of its vanishing-point, V^x, for finding the plan of hips and valleys and other lines lying at 45° with the principal lines. Further illustrations of this may be found in Plates II. and III.

V^x

If the principal horizontal lines of the perspective plan lie at 45° with the ground-line, as in Fig. 138, one set of the right-angles in which they meet will be bisected by lines drawn to the Centre, and the others by lines drawn parallel to the Horizon. The hips, also, on the right and left of the roofs, will be parallel to the picture, and will be drawn parallel to the horizons of the planes in which they lie, as shown in the figure.

458. Dimensions taken by scale upon the ground-line may be transferred to lines lying in the horizontal plane and parallel to the plane of the picture, and accordingly parallel to the ground-line, by drawing lines to any point on the Horizon as a vanishing-point of parallel lines. The fence in Fig. 137, B, is drawn in this way.

Lines parallel to *g l*.

459. As many different perspective plans may be made as the complexity of the subject may seem to require, and they may be above or below the picture, as may be most convenient, as in Plates III. and IV. This is illustrated also in Plate XXVII., where Fig. 140 shows three perspective plans, and Fig. 137 two. The work upon them may be done all at once, or from time to time during the progress of the drawing, as may be preferred. All the details of the plans of every part may thus be put into perspective. But it is not of course necessary to complete the plan of any parts that cannot be seen. In general it suffices to make the plan of the two sides that show, and of such more remote portions as are visible above these sides.

Several perspective plans.

460. The perspective plan being made, or at any rate fairly begun, the drawing itself may be commenced. The perspective of the object itself lies directly above the plan, but how far above depends upon the relative altitude of the object and of the spectator. The points on a level with the eye will always, of course, be seen on the Horizon. So much of the object as is above the eye, perhaps the whole of it, will appear above the Horizon; whatever is below the eye will be drawn below the Horizon. The *starting-point*, that is to say, the lower end of the front corner, will lie directly above the corresponding point in the perspective plan, and as far below the Horizon, by scale, as the

The perspective.

The position of the object, vertically.

The starting-point.

spectator's eye is supposed to be above the point itself, as at *c*, Fig. 137. The same perspective plan will serve to make several views of the same object, taken at different levels, above or below the starting-point, as is done in Figs. 4, 5, and 6, Plate II.

461. The perspective plan, drawn in the plane of the picture, suffices to determine all horizontal dimensions; that is to say, the position of all vertical lines.

The line of vertical measures.

The position of horizontal lines is determined by laying them off upon the initial-line or trace of some vertical plane, as a line of vertical measures. When the nearest corner touches the plane of the picture it is generally used for this purpose. This line is then at once the initial-line of the right-hand vertical plane, R Z, and of the left-hand vertical plane, L Z, and serves as a line of vertical measures for both, as at $v\,v$ in Figs. 137, B, and 138, B. The scale employed is the same as that used upon the ground-line for determining the horizontal dimensions of the perspective plan, since all lines in the plane of projection are drawn to the same scale (94).

Several such lines.

462. But any plane occurring in the object may be prolonged until it cuts the perspective plane, and have a line of measures of its own, in its own initial-line, as at $v'\,v'$ in Fig. 137, B, which serves as an independent line of measures for the end of the wing. The vertical dimensions taken upon these lines of measures may be transferred directly to any vertical line which lies in this vertical plane, and which is accordingly parallel to the line of

vertical measures, by means of the vanishing-points V^R and V^L. In this way is determined the position of all the horizontal lines in Figs. 137, B, and 138, B, the vertical lines erected from the corresponding lines in the perspective plans serving to determine their length.

463. Moreover, just as such dimensions were transferred to lines lying in the ground-plane, but inclined to the picture, by means of a point of distance upon the Horizon, so dimensions taken by scale upon a vertical line of measures may be transferred to lines that lie in the vertical plane and are inclined to the plane of the picture, by means of points of distance taken upon the horizon of the vertical plane in which they lie (113). This is illustrated in Fig. 12, Plate IV., where the heights of the gable are set off upon the vertical line through the corner of the house.

Points of distance upon vertical horizons.

464. If any part of the object advances in front of the principal vertical planes, or, in plan, in front of the principal lines of the perspective plan, as is the case with the wing of the building shown in Fig. 137, its plan can be drawn in perspective by prolonging the leading perspective lines *in front* of the perspective plane, as is shown in Fig. 139. In this figure the dimensions to be set off upon this part of a left-hand line, L (or of a right-hand line, R), are set off upon the ground-line to the right of the initial-point instead of to the left (or to the left instead of to the right); and in transferring them to the perspective line they are brought forward away from the

Constructions in front of the perspective plane. Fig. 139.

point of distance, instead of being carried backward toward it, as before.

The length of the wing of the building in Fig. 137 is ascertained in this way: The dimension R^2, taken from the elevation above, is laid off upon the ground-line *to the left* of the point I, the initial-point of the perspective line, R, and is transferred to the prolongation of that line in front of the plane of projection and below the ground-line by means of the point of distance, D^R, as in Fig. 139. See also Fig. 15, Plate V. (109).

Another way of drawing such objects, or parts of an object, is shown in the second perspective plan at the bottom of the same figure. The point *a*, where the wing joins the main building, having been ascertained as before, by measuring off upon the principal left-hand line the distance, L^1, a right-hand line, directed towards the right-hand vanishing-point, V^R, is drawn through the point *a* until it intersects the ground-line at *b*, its initial-point. If now a second line be drawn through *a*, directed upon D^R, the right-hand point of distance, and cutting the ground-line at *d*, the distance, *b d*, intercepted upon the ground-line, will be the real length of the line *a b*, and the real length of the wing, R^2, may be laid off upon the ground-line from *d* and transferred to the line *a b* by means of D^R, as shown.

If parts of the object to be drawn are advanced not only in front of the principal planes, but in front of the plane of projection, as often happens with the cornices of buildings, and with steps and platforms, as is shown in Fig. 138, they may be put into the perspective plan

by the methods just described. In this case the points and lines in which the several lines and planes cut the plane of projection are their initial-points and lines, and may be set off by scale. The points *a a*, at which the eaves of the building in the figure, for example, pierce the plane perspective, are equally far above the Horizon, and on a level with the top of the corner between them.

465. Fig. 140, which is a view of the spire of the church of St. Stephen's, Walbrook, illustrates the use of several perspective plans, and also the advantage of taking a perspective plane considerably in front of the object instead of in contact with it. The extension of the right-hand and left-hand vertical planes of the tower until they cut the perspective plane gives five initial-lines, or lines of vertical measures, on each side, all of which are quite outside of the picture, instead of one in the middle of it, as is the case when the front corner is taken as the line of measures. These are lettered I^R, $I^{R'}$, etc., I^L, $I^{L'}$, etc. respectively. This entirely frees the picture from constructive lines.

Fig. 140. Two lines of vertical measures.

Setting the object some distance behind the perspective plane of course makes its perspective smaller, but this may be met by setting off the dimensions upon the lines of measures at a larger scale, which, when the position of the vanishing-points remains unchanged, is equivalent to moving the plane of the picture nearer to the object itself. In Fig. 140, the scale employed at B for horizontal distances upon the ground-line, and for vertical

dimensions in the lines I^R, I^L, etc., is double that of the elevation at A, from which the dimensions are taken.

466. The same result may be produced by employing scales of vertical measures beyond the object, in accordance with the theory of small-scale data discussed in a previous chapter (339). If these are set up half-way between the initial-lines, or lines of vertical measures, and their corresponding vanishing-points, the scale to be used will be half as large, as in the figure at $v\,v$, $v'\,v'$, etc., where the heights set off are the same as in the elevation alongside.

Small-scale data.

If both scales of height are used, as in the figure, one on the right of the picture and another, at half the scale, on the left, the use of the left-hand vanishing-point, V^L, may be dispensed with, the perspectives of the horizontal lines being put in by drawing lines between the corresponding points on the two scales.

The vanishing-point unnecessary.

467. In the largest of the perspective plans employed in the figure, below the picture, advantage is taken of the fact that the two sides of the tower are exactly alike, to dispense also with the use of the point D^R. The points ascertained upon the left-hand side by means of the left-hand point of distance, D^L, are transferred to the right-hand side by means of the diagonal line directed towards V^X, the "vanishing-point of 45°," in accordance with the principle illustrated in Fig. 6, Plate II. (61).

The method of diagonals.

This plan illustrates also the principle of Auxiliary Horizons discussed in section 365. But instead of sinking the perspective plan in order

Auxiliary Horizons.

to prevent the angles of intersection from being too acute, and accordingly putting the ground-line four or five inches lower down, an auxiliary Horizon, H′ H′, is drawn in four or five inches above the real Horizon, the ground-line being retained, and the lines of the perspective plan are directed to the vanishing-points and points of distance found upon this new Horizon, as in Plates XXII. and XXIII.

468. Lines lying in an oblique plane can be measured off by means of a scale taken upon a line of measures which is the initial-line of the plane in which they lie, just as well as lines lying in horizontal or in vertical planes. This line will of course be parallel to the horizon of the plane drawn through the vanishing-points of two of its elements (398). If the lines are parallel to the picture they will be parallel both to this horizon and to this trace or line of measures, and dimensions taken upon the line of measures may be transferred to the perspective line by drawing parallel lines to any point upon the horizon as a vanishing-point. Dimensions taken upon the line of measures may be transferred to lines lying in the oblique plane and inclined to the picture, as before, by means of a point of distance taken upon the horizon of the oblique plane. See Fig. 15, Plate V. (111).

Oblique planes.

469. In the case of lines that lie at the intersection of two planes, it is a mere matter of convenience whether they shall be treated as lying in one plane or in the other. Either plane will

Choice of lines of measures.

do, its initial-line serving as a line of measures. Its horizon will contain the vanishing-point of the line in question, and its points of distance (412). All the points of distance of a line will be at the same distance from its vanishing-point, since the *locus* of its points of distance is a circle, of which the vanishing-point is the centre (423), and the radius the optical line.

470. These processes suffice not only to give the perspective of every point and line the position and direction of which is known, but to furnish several methods by which they can be determined. Whether one or another of the methods shall be employed in a given case is a matter of discretion, in which the judgment and experience of the draughtsman must guide his choice. A line, for instance, may be determined in direction either by fixing the position of the points in it, or by finding its vanishing-point. It is sometimes more convenient to do one, sometimes the other. Whichever is employed, the other may be used to test the accuracy of the result.

Choice of methods.

471. In completing a perspective drawing, many special devices may be employed to abbreviate labor. Of these the most important are the different ways of dividing lines in a given ratio, the different ways of casting shadows by natural or by artificial light, the use of points of half-distance or quarter-distance, and the various other devices for bringing the work within small limits, Mr. Adhémar's scheme for avoiding the difficulties experienced in drawing distant

Special devices.

objects, by the use of inclined perspective plans, with the suggested modifications, the employment of lines already existing as horizons of auxiliary planes, and the special processes to be followed in putting circular arcs into perspective, with the practical adjustments to be made in the results. It is not necessary again to go into these details of procedure.

472. It is, however, worth while to say that there are some mechanical devices not mentioned in the previous pages which are of service when vanishing-points are inconveniently far off. One of these is the employment of wooden or brass arcs, fastened to the drawing-board, upon which a T-square armed with pegs moves so as to direct its upper edge always to the centre of the arc, as a vanishing-point. Another way to effect the same thing is to cut a curve upon the handle of the T-square, and move it upon pins driven into the drawing-board. See Fig. 141.

Mechanical aids.

473. A third device is to cover the paper with a network of lines, drawn in pencil, and directed towards the principal vanishing-points, as a guide in sketching. A useful variation of this is a paper carefully ruled in ink, to be used in sketching upon paper laid over the lines, and thin enough for them to be seen through it.

474. It may be added that the simplest way, in practice, to obtain the position of the station-point in plan, and the direction of the principal right-hand and left-hand lines, R and L, when the principal vanishing-points, V^L and V^R, have been assumed (439), is to drive pins into the drawing-board at the vanishing-points, and then

to hold two T-squares at right angles to one another, and move them upon these pins. The right angle in which they meet will of course sweep the board in a semicircle, and if arrested at the point where the two edges of the T-squares have the required direction, will give the station-point.

APPENDIX.—NOTES.

It may perhaps help to make the main facts of Oblique or Two Point Perspective and of Parallel or One Point Perspective seem as simple as they really are, if they are presented in their simplest form. Plate XXVIII. accordingly shows several typical buildings, with gables and dormers, hips and valleys, drawn in both of these ways, with perspective plan, perspective, and bird's-eye view. The lines and planes in the plan and in the bird's-eye view, and their vanishing-points and horizons, are lettered in accordance with the notation used in this book. See Note VI.

Note I. — *Oblique Perspective.*

Fig. 142 shows a building in Two Point Perspective, the angle a at the top of the diagram showing the *attitude*, or the angle which the building makes with the plane of the picture. The position of the Station Point, in front of the picture, is shown in its revolved position at S', the right and left hand vanishing-points being taken at V^R and V^L. The position of S' is determined by lines drawn parallel to those at a, from V^R and V^L. This gives D^R and D^L, the points of distance, and the position of V^C, the Centre, and of V^X and V^Y, the vanishing-points of diagonals, in the plane of the picture, upon the Horizon. The Optical lines, R^0 and L^0, by which these are determined, lie really in a horizontal plane extending from the horizon H R L to the Station-Point, in the *air* in front of the picture. This plane is shown revolved into the plane of the picture about the horizon H R L.

The vertical horizons H R Z and H L Z being then drawn, the vanishing-points V^M, $V^{M'}$ and V^N, $V^{N'}$ are found upon them by laying off the angle β at D^R and D^L. The inclined horizons H R N

and H R N′, H L M and H L M′, are then drawn, and the vanishing-points V^P and $V^{P'}$, V^Q and $V^{Q'}$, found at their intersection. Just as V^P and the $V^{P'}$ come on the horizon H X Z, above and below V^X, so the vanishing-points V^Q and $V^{Q'}$ are found in the horizon H Y Z, above and below V^Y.

It is to be noticed that the position of V^Y, and hence of H Y Z, V^Q and $V^{Q'}$, can be got by drawing a construction line, otherwise meaningless, through V^M and V^N or through $V^{M'}$ and $V^{N'}$, and that in like manner V^X lies where a line through V^M to $V^{N'}$, or $V^{M'}$ to V^N, cuts the Horizon. It is sometimes convenient to determine these points in this way.

NOTE II.—*45° Perspective.*

Fig. 143 exhibits the characteristic peculiarities of 45° Perspective : the diagram is symmetrical both about the Horizon H R L and about the Vertical Horizon H X Z ; V^Y, V^Q and $V^{Q'}$ lie at an infinite distance in the plane of the picture ; H R N is parallel to H L M′ and to Q, and H R N′ to H L M and to Q′; V^X and V^C coincide, both being at the Centre ; and D^R and D^L are equidistant from it, being about two-fifths of the way from V^C to V^R or V^L.

It is to be noticed that D^R and D^L come just where the corners of an octagon would come if cut from a square whose side was as long as the line from V^L to V^R.

These conditions make it easier to draw a building when it makes an angle of 45° with the picture than when it is in any other position.

These conditions—viz.: that in 45° perspective V^X coincides with V^C and comes half way between V^R and V^L, and that while one diagonal of a horizontal square is directed toward this point, the other, being directed toward the infinitely distant V^Y, is parallel to the Horizon, that is to say, horizontal—make it very easy to put any number of such squares into perspective. As the plans of most buildings are laid out on a modulus of equal parts, that is to say, are composed of squares, it is accordingly very easy to draw a perspective plan of such buildings and hence to draw the per-

spectives of the buildings themselves. The scale of the vertical lines is the same at any point as that of the horizontal diagonal of a square at that point, the nearer squares coming out, of course, larger than the more distant ones.

Let us suppose that, as in Fig. 144, we have a building 100 feet long, 50 feet wide, with a tower 50 feet square and a porch 25 feet by 50, 25 feet high up to the eaves, 50 feet to the ridge, and 100 feet to the top of the tower, and that the porch walls are half as high as those of the church.

V^R and V^L being taken as far apart as is convenient upon a line taken for the Horizon, and V^X half way between them, the three whole squares and one half square of the perspective plan can be drawn anywhere below it at any convenient scale, the sides of the squares being directed toward V^L and V^R, one of the diagonals toward V^X, and the other drawn horizontally. These may be taken to represent the perspective of squares that measure 50 feet on a side. The perspective of the building can then be drawn directly above the plan, the lower end of the nearest front corner being taken just over the corner of the nearest square, and the other vertical lines drawn above the other corners. The vertical line that passes through the gable is drawn through the middle of the left-hand side at a point determined by drawing a line through the middle of the building in the perspective plan below. The height of 50 feet is then to be laid off on the front corner of the building. The length of 50 feet at that point is determined as follows: From the point below it, a, in the perspective plan, is drawn the diagonal of a third square from a to b. This diagonal is of course drawn to the scale at which the front corner of the building is to be drawn, being equally distant from the plane of the picture. It is the diagonal of a square whose side is 50 feet. The length of that side, that is to say, the length of a line 50 feet long at that point, is found by erecting upon the line $a\,b$, as a hypothenuse, a half-square $a\,b\,c$. The line $b\,c$ is obviously a line 50 feet long, at the scale in question.

If now in the perspective above we lay off this distance upon the front corner from a' to a'', half this height will give the required 25 feet for the height of the eaves, and the other half the addi-

tional 25 feet for the height of the ridge. Doubling this line gives the height of the tower, and halving it the height of the porch walls. These are all the *data* required for completing the drawing.

Here, as in most cases, the scale of the perspective is on the whole smaller than the scale at the nearest corner, and as it is generally convenient to make the scale at this corner the same as that of the orthographic elevation, the result is that the details have to be drawn considerably smaller in the perspective than in the geometrical elevations. This can be obviated by taking the Perspective Plane somewhere in the middle of the building, or, as is sometimes done, using one of the further corners instead of the front corner as a line of vertical measures. In this case the half of the building which is seen comes in front of the perspective plane instead of behind it, and the details are drawn on a scale somewhat larger than that of the elevations instead of smaller.

The relations just described make it easy to effect this in the case of a square building, such as a monument or tower, set at an angle of 45°.

This is illustrated in Fig. 145, which shows such a structure in elevation at A. At B it is shown in diagonal elevation, a view which is often worth the trouble to make in order to see how a design will appear in its most unfavorable aspect. It is of course easily made by laying off upon horizontal lines the "diagonals" of the horizontal distances from the centre instead of the actual distances. If now at the points thus ascertained only the vertical elements of the outline are drawn, as shown at C, it is easy to construct the required perspective drawing within the limits thus determined. All that is necessary is to draw a Horizon at any convenient height and to take upon it three equidistant points V^L, V^X, and V^R. Pains must be taken not to have V^X come exactly on the axis of the building. If now from the extremity of the vertical outlines thus determined lines are drawn from V^L and V^R, the points where they intersect will determine the perspective of the front corner. The result will be a true perspective of the building, and the outline will be drawn at the same scale as the elevation, as appears in the figure.

Note III. — *Parallel Perspective.*

Fig. 146 exhibits in a somewhat simpler manner than Plate VI. the distinctive features of One Point, or Parallel, Perspective. Here again the diagram is symmetrical about both the vertical and the horizontal horizons that pass through the centre, V^C. The vertical planes are either parallel or normal to the plane of the picture, and the inclined planes of the roof are either normal, as in the case of the planes C D and C S, or, as in the case of the planes K A and K A′, they have their horizontal element parallel to the picture, so that their horizons, H K A and H K A′, are horizontal and parallel to the Horizon H C K (or H R L); the diagonals X and Y coincide in direction with R and L, and the lines of the hips and valleys P and P′, Q and Q′, with M and M′, N and N′.

Note IV. — *The Inverse Process.*

Besides the three devices mentioned in the text there are three other ways of obtaining an elevation from a perspective drawing or photograph. The problem in every case is to find D^R and D^L when V^R and V^L are given, some third point being also supplied. It is plain from the relations shown in Fig. 139 A, Plate XXVII., and again in Fig. 148, that if V^R and V^L are given, and also either S′, V^C, D^R, or D^L, the other three points can easily be obtained.

I. The first case is that in which the Centre V^C is indicated by some object given in Parallel Perspective. This case has already been illustrated in Fig. 111 A, Plate XXIV., and discussed in Section 384, page 250.

The whole procedure is shown in Fig. 148, Plate XXIX. S′ is obtained from V^C, and D^R and D^L from S′. Lines drawn from D^R determine, upon the Ground Line, T R L, the points *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*, which give the real width of the piers and arches on the right-hand side of the building, on the scale of the nearest corner. In like manner lines drawn from D^L determine the points *i*, *j*, and *k*, which give the horizontal dimensions of the left-hand side, on the same scale. The points *l*, *m*, *n*, *o*, *p*, *q*, *r*, *s*, *t*, *u*, and *v*, taken from points on the front corner itself, give the vertical

dimensions. From these the two elevations can be constructed, as is done in Fig. 154.

II. The second case is that in which V^X is indicated by a horizontal square, the diagonal of which is directed to V^X as its vanishing-point. In Fig. 149 such a square is found by taking equal distances upon the right and left hand cornices, each including six modillions. By drawing a circle of which the line $V^L V^R$ is the diameter, and drawing a line from the top of the circle through V^X, the position of S' is determined, as has been already explained in Section 385. D^R and D^L are then easily found, as in the preceding example.

III. The third case is that in which V^P is given by the hips and valleys of which it is the vanishing-point, Fig. 150. V^X is found upon the Horizon directly below it, and the problem becomes the same as in the previous case.

IV. The fourth case is that in which a square or half-square occurs in a vertical plane, or is given by a circle or half-circle, as, for example, by a circular window, or by a semicircular arch, as in Fig. 151. The diagonals of these parallelograms have their vanishing-points at V^M1 or $V^M\frac{1}{2}$, on H R Z, the vertical horizon of the plane in which they lie. The true inclination of these lines is known, being the angle which they make with the horizontal plane, the angle which we have called β. D^R is easily found by laying off the complement of this angle at V^M1 or $V^M\frac{1}{2}$, and drawing a line to the Horizon.

The simplest way to get D^R is to lay off on the Horizon, from V^R, the distance $V^R V^M1$, or twice the distance $V^R D^M\frac{1}{2}$, that is to say, to revolve into the plane of the picture the base of the optical triangle, $S V^R V^M1$, or $S V^R V^M\frac{1}{2}$.

If the vertical square or half-square were on the left-hand side of the building, V^N1 or $V^N\frac{1}{2}$ would be employed to obtain D^L.

The point, V^M, on H R Z, indicates, as usual, the vanishing-point of the lines of the gable and of the steepest line of the roof.

Of course any other feature whose shape was known would give the angle β as well as a square or semicircle.

V. The fifth case gives a point of distance, D^R or D^L, directly. Fig. 152. Here also a vertical square occurs in a vertical perspective plane. If a horizontal line is drawn from one end of its

vertical side as long as that side, this line and the perspective of the horizontal side will be the perspective of the sides of an isosceles triangle lying in a horizontal plane. The vanishing-point of the base of this triangle will be the required point of distance. This isosceles triangle is half of a horizontal square, and its base is the diagonal of the square. S′, the revolved position of the Station point, and the other point of distance are then easily found.

If the isosceles triangle, as in the figure, lies so near the Horizon as to be very much foreshortened, the angles coming out too acute for accurate draughtsmanship, it can be re-drawn at a lower level, as shown.

This is the same principle that is illustrated in Plate XXIV., Fig. 113, and discussed in Section 386.

VI. The sixth case, illustrated in Fig. 153, also gives a point of distance directly, and again employs a vertical square, using it just as the horizontal square was used in Fig. 152. The point of distance of R, the horizontal side of the square, is again the first point determined. But as the square of which this line is a side is now a vertical square, this point of distance, instead of being on the Horizon, H R L, lies in the vertical horizon, H R Z. Either half of the square is an isosceles triangle, the base of which is the diagonal of the square, and the vanishing-point of this diagonal, in the vertical horizon H R Z, is accordingly the vanishing-point of the base, or point of distance, D^R. Lines drawn from this point through the lower ends of the piers determine upon the vertical trace T R Z the points *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*, which give the real width of the piers and arches on the right-hand side of the building, just as these were determined on the Ground Line, T R L, in Fig. 148.

D^R on the Horizon H R L is of course just as far from V^R as is D^R on H R Z, both distances being equal to the length of the Optical line R^O. Both points are situated on a circle of which V^R is the centre and R^O the radius, and which is the *locus* of D^R. S′, D^L, V^C, and V^X can then easily be found, if wanted.

Note V. — *Shadows by Artificial Light.*

Fig. 155 illustrates the proposition that if an auxiliary line, or ray, be drawn through a source of divergent rays as an Apex, parallel to a given system of lines, the shadows of those lines upon any plane will diverge from the point where the auxiliary line pierces the plane. This has already been illustrated in Plates XVII., XVIII., and XIX. In the figure the rays of light diverge from A as their Apex, the shadows of the vertical lines Z from A_Z, and the shadows of the inclined lines M from A_M.

In Fig. 155 the source of light, A, is so low down and V^M, the vanishing-point of the inclined lines, is so high up, that the point A_M, where the auxiliary line strikes the horizontal plane, lies between the ground line and the Horizon. But it may happen that the source of light is so high above the ground plane, or the inclination of the given lines so slight, that the auxiliary line pierces the plane of the picture before striking the ground plane. In that case the Apex A_M will be in front of the plane of the picture and may even be so far in front of it as to be behind the spectator.

This is illustrated in Fig. 156. In the orthographic plan below are shown the plane of the picture, *p p*, and the station-point in front of it, S. The optical line C^0, normal to the picture, gives the position of the Centre V^C, and the optical line R^0, which is the horizontal projection of M^0, the optical line of the given system, gives the position of V^R and the position of D^R. In the perspective above, the angle laid off at D^R on the Horizon gives the position of the vanishing-point V^M. If now the position of the source of light is assumed to be at A, and its projection on the ground plan to be at A_Z, the auxiliary line M can be drawn through V^M and A, and its projection R on the ground plane through V^R and A_Z. R and M will both lie in a vertical plane R Z and their vanishing-points will be in its horizon, H R Z, and their initial-points in its trace, T R Z. R will pierce the plane of the picture at its initial-point I^R, where it cuts the ground line, and if produced in front of the picture, as shown in the orthographic plan below, it will lie in the ground plane, in a direction

parallel to the optical line R^0. The auxiliary line M will, if it also is continued in front of the picture, descend from its initial-point I^M until it meets the line R at A_M. This, as before, is the apex from which diverge the shadows of the lines M cast upon the ground plane.

The point A_M is easily found, since the trace T R Z is one side of a vertical right triangle, extending in front of the picture, of which the line R, produced, is the base and the line M, produced, is the hypothenuse. The length of the line R is found by revolving this triangle into the plane of the picture around the trace T R Z, as shown, making the angle at the base equal to β. That is to say, the hypothenuse M in its revolved position is drawn parallel to the line $D^R\ V^M$, which is M^0 in its revolved position.

It is to be observed that the auxiliary line M taken through A really ascends from I^M to V^M, since it makes the angle β with the ground plane. But it seems to descend, the perspective of its upper end V^M coming lower down on the paper than that of its lower end I^M. This often happens when one looks up at an ascending line at an angle steeper than that of the line itself.

The apex A_M being behind the spectator, the shadows of the lines M, cast upon the ground plane, which really diverge from this apex, seem to converge upon a false apex A_M', in front of the spectator.

This false apex may be found, as in Fig. 80, Plate XVII., by passing through the apex A_M two horizontal lines diverging from it, one normal to the plane of the picture and the other passing directly under the station-point. These lines lie in the ground plane. Calling them respectively C and R′, the portions in front of the plane of the picture as shown in the orthographic plan will pierce the plane of the picture at I^C and $I^{R'}$, on the ground line, and the perspectives of the portions behind the picture will extend from I^C to V^C and from $I^{R'}$ to $V^{R'}$, respectively. These lines point to the false apex A_M'.

But as M and R also diverge from the apex A_M as their real apex, the perspectives of these lines also are directed toward the false apex, and A_M' can be found at once, without making any orthographic plan at all. It is at the intersection of M and R, extended, just as the real apex is found at the intersection of M and R in Fig. 81, *h*, and in Fig. 155.

If A, the apex of diverging rays, is so situated that the apex of shadows, A_M, though in front of the picture, is neither in front of the spectator nor behind him, the line C being just as long as the Axis C^o, then the perspectives of the diverging shadows will be parallel, just as the lines 2, 7, and 4 and lines 5, 8, and 10 are in Fig. 77, and their false apex will be at an infinite distance. This is illustrated in Fig. 157. It will be observed that under these conditions the apex A comes just as far above the ground line as the vanishing-point V^M is above the Horizon, and that M, R, and the shadows of M on the ground plane are all parallel also.

Under these circumstances R′ becomes parallel to the picture and is replaced by K.

NOTE VI. — *The Perspective Alphabet.*

The Notation employed in this book is exhibited in the following Table. The words in parenthesis explain the significance of the letters, and make it easy to remember. Most of these letters stand for lines. Those which indicate points are marked thus ×.

	A, A′	(*Altitude*)	**Inclined lines, sloping up or down, in vertical normal planes.**
×	**A**		**An Apex. The point at which convergent lines really meet.**
×	**A′**		**A False Apex, or point toward which divergent lines seem to converge.**
×	**B, B′**		**Other apexes and false apexes.**
	B	(*Base*)	**Lines parallel to the base of the isosceles triangles, having the point of distance for their vanishing-points. V^B = D.**
	C	(*Centre*)	**Normal Lines, having their vanishing-points at the Centre V^C.**
	D	(*Dexter*)	**Lines parallel to the picture sloping down to the right.**
×	**D**	(*Distance*)	**A point of distance, which is a point at the same distance from the vanishing-point that the vanishing-point is from the station-point. S V = V D.**
	D^R, D^L, D^M, etc.		**Points of distance of the systems R, L, M, etc.**
	E, F, G		**It is sometimes convenient to use these letters to designate the main lines in Three Point Perspective.**
	H	(*Horizon*)	**A Horizon, the infinitely distant line where the parallel planes of any system seem to meet, or its perspective in the plane of the picture.**

	H', H'', etc.		Horizons of auxiliary planes.
	H R L, H R Z, etc.		The Horizons of the planes R L, R Z, etc. . . . ; or their perspectives. Horizons are indicated thus : —— . —— . —— . —— . ——.
	H H = H R L		*The* Horizon; the horizon of horizontal planes.
×	I	(*Initial*)	An Initial point; the point where an inclined line behind the plane of the picture pierces it; the first or nearest point of such a line.
×	I^R, I^L, etc.		The Initial points of the lines R L, etc.
	K	(*Instead of* H)	Horizontal lines parallel to the plane of the picture.
	L, L'	(*Left hand*)	Horizontal lines inclined to the plane of the picture and going back to the Left.
	M, M'		Oblique lines going up, or down, and back to the Right.
	N, N'		Oblique lines going up, or down, and back to the Left.
	O	(*Optical*)	This letter designates an optical line, that is, the element of a system which passes through the eye, or station-point. It is in front of the picture, which it pierces at the perspective of the vanishing-point of the system. Its length is the distance of the vanishing-point from the station-point.
	R^O, L^O, etc.		The optical lines of the systems R, L, etc.
	C^O		The Axis; the optical line of the system C, normal to the plane of the picture. Its length is the distance of the eye from the nearest point of the picture, that is, from the Station-point to the Centre.
	P P		The Perspective plane, or Plane of Measures, situated near the object.
	p p		The Plane of the Picture, situated near the spectator, parallel to the Perspective Plane.

(When the object is small, or is a model of the real object, P P coincides with *p p*.)

	P, P'		Oblique lines, such as hips and valleys, lying in or parallel to the intersections of oblique planes, and nearly normal to the picture.
	Q, Q'		The same, when nearly, or quite, parallel to the picture.
	R, R'	(*Right hand*)	Horizontal lines inclined to the plane of the picture and going back to the Right.
	S	(*Sinister*)	Lines parallel to the picture, sloping down to the Left.
×	S	(*Spectator*)	The Station-point, or position of the eye, in the air in front of the picture.

×	S′, S″, etc.		The position of the Station-point when revolved into the plane of the picture.
	T	(*Trace*)	A Trace, or initial-line; the line in which an inclined plane behind the plane of the picture intersects it. Traces are indicated thus: —— —— —— —— —— ——.
	T′, T″, etc.		Auxiliary traces.
	T R L, T R Z, etc.		The traces of the planes R L, R Z, etc.
	T	(*T-square*)	Lines normal to a given system of planes; their axes.
	T^{RL}, T^{LM}, etc.		Lines normal to the planes R L, L M, etc.
×	V		A vanishing-point; the infinitely distant point where the lines of any system seem to meet; or its perspective in the plane of the picture.
	V^R, V^L, V^M, etc.		The vanishing-points of the systems R, L, M etc., or their perspectives.
×	V^X		"The Vanishing-point of diagonals," or "of Forty-Five Degrees."
×	V′, V″		Vanishing-points of auxiliary lines.
	X	($\boxtimes$)	"Diagonal" lines, or "45°" lines; horizontal lines nearly or quite normal to the picture, and in the direction of one diagonal of a square, the sides of which, R and L, are at right angles with one another.
	Y		Similar lines in the direction of the other diagonal, and nearly or quite parallel to the picture.
	Z	(*Zenith*)	Vertical lines, parallel to the plane of the picture, and having their vanishing-points in the Zenith and Nadir.
	Z Z = C Z		The principal vertical horizon; the horizon of vertical planes normal to the picture.

The Letters J, U, and W are not employed.

G L, or *g l* — These letters are used to designate the Ground Line, or trace of the ground plane.

Planes are designated by the letters which indicate their two principal elements, viz.: The horizontal element and the line of steepest slope; as R N, C S, L Z, etc.

Finite lines are designated by small capitals: R, L, C, Y, Z, etc.

Points are marked thus ×, and are designated by small letters, thus: *a*, *b*, etc., *m*, *n*, etc.

Letters denoting lines revolved into the plane of the picture are enclosed in parentheses, thus: — (R^0).